CHAPTER ONE

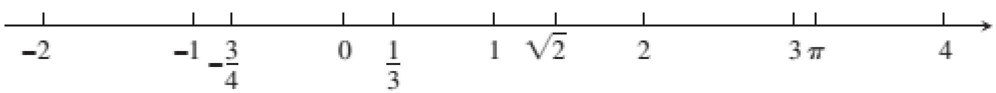
1.1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers: are numbers that can be expressed as decimals, such as

$$-\frac{3}{4} = -0.75000\ldots$$
$$\frac{1}{3} = 0.33333\ldots$$
$$\sqrt{2} = 1.4142\ldots$$

The real numbers can be represented geometrically as points on a number line called the real line.



THE ALGEBRAIC PROPERTIES

The real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic.

There are three special subsets of real numbers.

1. The natural numbers, namely 1, 2, 3, 4

2. The integers, namely

$0, \pm 1, \pm 2, \pm 3, \ldots$

3. The rational numbers, namely the numbers that can be expressed in the form of a fraction , where**(m and n)** are integers and $n \neq 0$,

Examples are:

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

INTERVALS

- A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all x> 6 real numbers *x* such that is an interval. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1and 1 (for example).

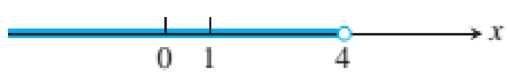
	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 : Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \ge 5$

(a)

$$2x - 1 < x + 3$$
$$2x < x + 4$$
$$x < 4$$



The solution set is the open interval $(-\infty, 4)$

(b)

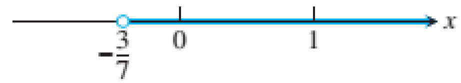
$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \qquad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \qquad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \qquad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \qquad \text{Divide by 7.}$$



(b)

The solution set is the open interval $(-3/7, \infty)$

$|-a| \neq -|a|$. $|-3| = 3$, whereas $-|3| = -3$.

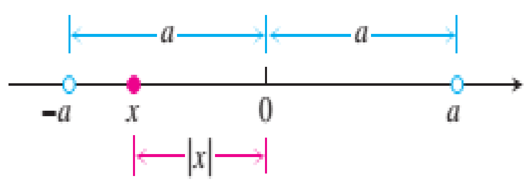
$|a + b|$ less than $|a| + |b|$. If a and b differ in sign,

$|a + b|$ equals $|a| + |b|$. In all other cases,

EXAMPLE : Illustrating the Triangle Inequality

$$|-3 + 5| = |2| = 2 < |-3| + |5| = 8$$
$$|3 + 5| = |8| = |3| + |5|$$
$$|-3 - 5| = |-8| = 8 = |-3| + |-5|$$

The distance from **(x to 0)** is less than the positive number **a**. This means that **x** must lie between (–a and **a**).



Absolute value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 Finding Absolute Values

$$|3| = 3, \quad |0| = 0, \quad |-5| = -(-5) = 5, \quad |-|a|| = |a|$$

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$ — The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

EXAMPLE: Solving an Equation with Absolute Values the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$2x - 3 = 7 \qquad 2x - 3 = -7 \qquad \text{Equivalent equations without absolute values}$$

$$2x = 10 \qquad 2x = -4 \qquad \text{Solve as usual.}$$

$$x = 5 \qquad x = -2$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

EXAMPLE: Solving an Inequality Involving Absolute Values Solve the inequality**.**

(a) $|2x - 3| \leq 1$ **(b)** $|2x - 3| \geq 1$

Solution

(a)

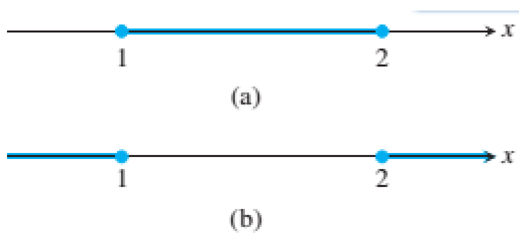
$$|2x - 3| \leq 1$$
$$-1 \leq 2x - 3 \leq 1 \quad \text{Property 8}$$
$$2 \leq 2x \leq 4 \quad \text{Add 3.}$$
$$1 \leq x \leq 2 \quad \text{Divide by 2.}$$

The solution set is the closed interval [1, 2] (Figure 1.4a).

(b)

$$|2x - 3| \geq 1$$
$$2x - 3 \geq 1 \quad \text{or} \quad 2x - 3 \leq -1 \quad \text{Property 9}$$
$$x - \frac{3}{2} \geq \frac{1}{2} \quad \text{or} \quad x - \frac{3}{2} \leq -\frac{1}{2} \quad \text{Divide by 2.}$$
$$x \geq 2 \quad \text{or} \quad x \leq 1 \quad \text{Add } \frac{3}{2}.$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b).



Solved Peroblems

5. $-2x > 4$

6. $8 - 3x \geq 5$

7. $5x - 3 \leq 7 - 3x$

8. $3(2 - x) > 2(3 + x)$

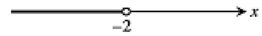
9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$

10. $\frac{6 - x}{4} < \frac{3x - 4}{2}$

11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$

12. $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

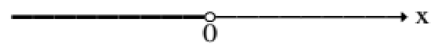
5. $-2x > 4 \Rightarrow x < -2$



6. $8 - 3x \geq 5 \Rightarrow -3x \geq -3 \Rightarrow x \leq 1$

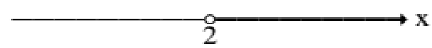
7. $5x - 3 \leq 7 - 3x \Rightarrow 8x \leq 10 \Rightarrow x \leq \frac{5}{4}$

8. $3(2 - x) > 2(3 + x) \Rightarrow 6 - 3x > 6 + 2x$
$\Rightarrow 0 > 5x \Rightarrow 0 > x$



9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6} \Rightarrow -\frac{1}{2} - \frac{7}{6} \geq 5x$
$\Rightarrow \frac{1}{5}\left(-\frac{10}{6}\right) \geq x$ or $-\frac{1}{3} \geq x$

10. $\frac{6-x}{4} < \frac{3x-4}{2} \Rightarrow 12 - 2x < 12x - 16$
$\Rightarrow 28 < 14x \Rightarrow 2 < x$



11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6) \Rightarrow 12(x - 2) < 5(x - 6)$
$\Rightarrow 12x - 24 < 5x - 30 \Rightarrow 7x < -6$ or $x < -\frac{6}{7}$

12. $-\frac{x+5}{2} \leq \frac{12+3x}{4} \Rightarrow -(4x + 20) \leq 24 + 6x$
$\Rightarrow -44 \leq 10x \Rightarrow -\frac{22}{5} \leq x$

Solve the equations (Absolute Value)in Exercises

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

Solution

13. $y = 3$ or $y = -3$

14. $y - 3 = 7$ or $y - 3 = -7 \Rightarrow y = 10$ or $y = -4$

15. $2t + 5 = 4$ or $2t + 5 = -4 \Rightarrow 2t = -1$ or $2t = -9 \Rightarrow t = -\frac{1}{2}$ or $t = -\frac{9}{2}$

16. $1 - t = 1$ or $1 - t = -1 \Rightarrow -t = 0$ or $-t = -2 \Rightarrow t = 0$ or $t = 2$

17. $8 - 3s = \frac{9}{2}$ or $8 - 3s = -\frac{9}{2} \Rightarrow -3s = -\frac{7}{2}$ or $-3s = -\frac{25}{2} \Rightarrow s = \frac{7}{6}$ or $s = \frac{25}{6}$

18. $\frac{s}{2} - 1 = 1$ or $\frac{s}{2} - 1 = -1 \Rightarrow \frac{s}{2} = 2$ or $\frac{s}{2} = 0 \Rightarrow s = 4$ or $s = 0$

Exercises with solution

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

19. $|x| < 2$ **20.** $|x| \le 2$ **21.** $|t - 1| \le 3$

22. $|t + 2| < 1$ **23.** $|3y - 7| < 4$ **24.** $|2y + 5| < 1$

25. $\left|\frac{z}{5} - 1\right| \le 1$ **26.** $\left|\frac{3}{2}z - 1\right| \le 2$ **27.** $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$

28. $\left|\frac{2}{x} - 4\right| < 3$ **29.** $|2s| \ge 4$ **30.** $|s + 3| \ge \frac{1}{2}$

31. $|1 - x| > 1$ **32.** $|2 - 3x| > 5$ **33.** $\left|\frac{r + 1}{2}\right| \ge 1$

34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Solution

19. $-2 < x < 2$; solution interval $(-2, 2)$

20. $-2 \le x \le 2$; solution interval $[-2, 2]$

21. $-3 \le t - 1 \le 3 \Rightarrow -2 \le t \le 4$; solution interval $[-2, 4]$

22. $-1 < t + 2 < 1 \Rightarrow -3 < t < -1$;
solution interval $(-3, -1)$

23. $-4 < 3y - 7 < 4 \Rightarrow 3 < 3y < 11 \Rightarrow 1 < y < \frac{11}{3}$;
solution interval $\left(1, \frac{11}{3}\right)$

24. $-1 < 2y + 5 < 1 \Rightarrow -6 < 2y < -4 \Rightarrow -3 < y < -2$;
solution interval $(-3, -2)$

25. $-1 \le \frac{z}{5} - 1 \le 1 \Rightarrow 0 \le \frac{z}{5} \le 2 \Rightarrow 0 \le z \le 10$;
solution interval $[0, 10]$

26. $-2 \le \frac{3z}{2} - 1 \le 2 \Rightarrow -1 \le \frac{3z}{2} \le 3 \Rightarrow -\frac{2}{3} \le z \le 2$;
solution interval $\left[-\frac{2}{3}, 2\right]$

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and show them on the real line.

35. $x^2 < 2$ **36.** $4 \le x^2$ **37.** $4 < x^2 < 9$

38. $\frac{1}{9} < x^2 < \frac{1}{4}$ **39.** $(x - 1)^2 < 4$ **40.** $(x + 3)^2 < 2$

41. $x^2 - x < 0$ **42.** $x^2 - x - 2 \ge 0$

35. $x^2 < 2 \Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$;
solution interval $\left(-\sqrt{2}, \sqrt{2}\right)$

36. $4 \le x^2 \Rightarrow 2 \le |x| \Rightarrow x \ge 2$ or $x \le -2$;
solution interval $(-\infty, -2] \cup [2, \infty)$

37. $4 < x^2 < 9 \Rightarrow 2 < |x| < 3 \Rightarrow 2 < x < 3$ or $2 < -x < 3$
$\Rightarrow 2 < x < 3$ or $-3 < x < -2$;
solution intervals $(-3, -2) \cup (2, 3)$

38. $\frac{1}{9} < x^2 < \frac{1}{4} \Rightarrow \frac{1}{3} < |x| < \frac{1}{2} \Rightarrow \frac{1}{3} < x < \frac{1}{2}$ or $\frac{1}{3} < -x < \frac{1}{2}$
$\Rightarrow \frac{1}{3} < x < \frac{1}{2}$ or $-\frac{1}{2} < x < -\frac{1}{3}$;
solution intervals $\left(-\frac{1}{2}, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right)$

39. $(x - 1)^2 < 4 \Rightarrow |x - 1| < 2 \Rightarrow -2 < x - 1 < 2$
$\Rightarrow -1 < x < 3$; solution interval $(-1, 3)$

Tutorials

- ❖ **proof of the triangle inequality** Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$\begin{aligned}
|a + b|^2 &= (a + b)^2 && (1)\\
&= a^2 + 2ab + b^2 \\
&\le a^2 + 2|a||b| + b^2 && (2)\\
&= |a|^2 + 2|a||b| + |b|^2 && (3)\\
&= (|a| + |b|)^2 \\
|a + b| &\le |a| + |b| && (4)
\end{aligned}$$

- ❖ Prove that $|ab| = |a||b|$ for any numbers a and b.

- ❖ Graph the inequality $|x| + |y| \le 1$.

- ❖ **a.** If b is any nonzero real number, prove that $|1/b| = 1/|b|$.

  b. Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any numbers a and $b \ne 0$.

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CHAPTER 2

1.3 Functions and Their Graphs

- Functions are the major objects we deal with in calculus because they are key to describing the real world in mathematical terms.
- This section reviews the ideas of functions, their graphs, and ways of representing them.

- **Functions; Domain and Range**

✓ The area of a circle depends on the radius of the circle.

✓ The distance an object travels from an initial location along a straight line path depends on its speed.

- In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”})$$

- In this notation, the symbol f represents the function. The letter x, called the **independent variable**, represents the input value of f, and y, the **dependent variable**, represents the corresponding output **value** of f at x.

> **DEFINITION Function**
>
> A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

- The set D of all possible input values is **called the domain** of the function.
- The set of all values of f(x) as x varies throughout D is **called the range** of the function.
- The range may not include every element in the set Y.

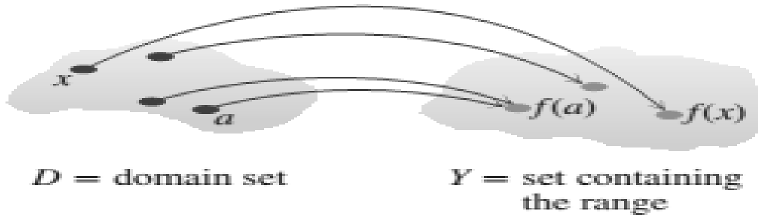


FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

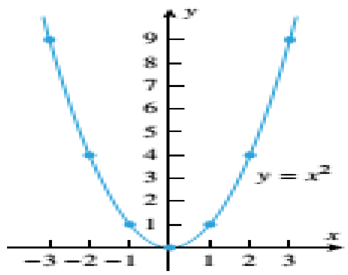
- If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

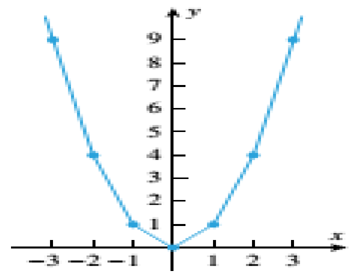
$$\{(x, f(x)) \mid x \in D\}.$$

> **F.1 DEFINITION.** The set of all solutions of an equation in x and y is called the ***solution set*** of the equation, and the set of all points in the xy-plane whose coordinates are members of the solution set is called the ***graph*** of the equation.

Example: Use point plotting to sketch the graph of ($\mathbf{y = x^2}$)Discuss the limitations of this method.

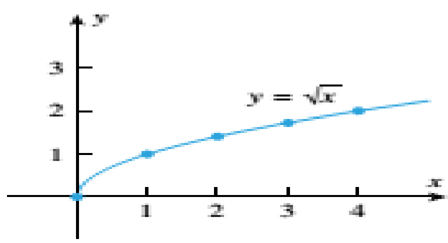
x	$y = x^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)
−1	1	(−1, 1)
−2	4	(−2, 4)
−3	9	(−3, 9)



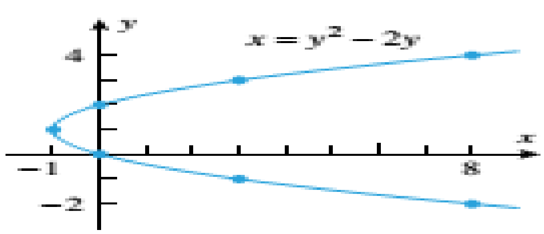


<u>Example</u>: Sketch the graph of($\mathbf{y = \sqrt{x}}$ **) .**
Solution
If ($x<0$), then ($\sqrt{x}$) is an imaginary number. Thus, we can only plot points for which **($x \geq 0$),** since points in the xy-plane have real coordinates. The graph obtained by point plotting.

x	$y = \sqrt{x}$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	$\sqrt{2}$	$(2, \sqrt{2}) \approx (2, 1.4)$
3	$\sqrt{3}$	$(3, \sqrt{3}) \approx (3, 1.7)$
4	2	(4, 2)



<u>Example</u> Sketch the graph of **($y^2 - 2y - x = 0$).**
Solution
In this case it is easier to express in terms of y, so we rewrite the equation as ($x = y^2 - 2y$) Members of the solution set can be obtained from this equation by substituting arbitrary values for y in the right side and computing the associated values of x.

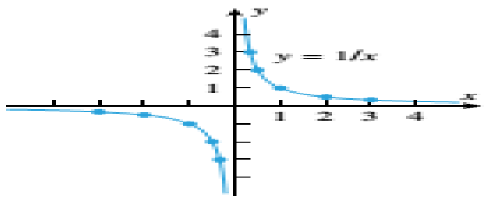


y	$x = y^2 - 2y$	(x, y)
−2	8	(8, −2)
−1	3	(3, −1)
0	0	(0, 0)
1	−1	(−1, 1)
2	0	(0, 2)
3	3	(3, 3)
4	8	(8, 4)

Example Sketch the graph of y=1/x.

Solution

Because **(1/x)** is undefined at x=0, we can only plot points for which **x ≠ 0**. This forces a break, called a discontinuity, in the graph at **(x = 0)**.



x	$y = 1/x$	(x, y)
$\frac{1}{3}$	3	$(\frac{1}{3}, 3)$
$\frac{1}{2}$	2	$(\frac{1}{2}, 2)$
1	1	$(1, 1)$
2	$\frac{1}{2}$	$(2, \frac{1}{2})$
3	$\frac{1}{3}$	$(3, \frac{1}{3})$
$-\frac{1}{3}$	-3	$(-\frac{1}{3}, -3)$
$-\frac{1}{2}$	-2	$(-\frac{1}{2}, -2)$
-1	-1	$(-1, -1)$
-2	$-\frac{1}{2}$	$(-2, -\frac{1}{2})$
-3	$-\frac{1}{3}$	$(-3, -\frac{1}{3})$

Greatest integer function is a piece-wise defined function.

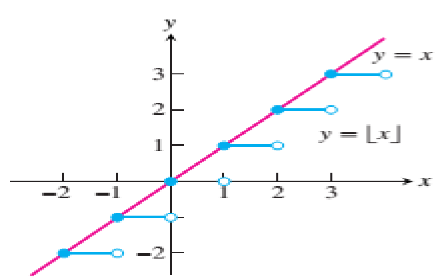
- ❖ The function, or rule which produces the "greatest integer less than or equal to the number" operated upon, symbol [x] or sometimes [[x]].
- ❖ The greatest integer function is a piece-wise defined function.
  - ❖ If the number is an integer, use that integer.
  - ❖ If the number is not an integer, use the next smaller integer.

Examples:

number	the greatest integer less than or equal to the number
x	[x]
4	[4] = 4
4.4	[4.4] = 4
-2	[-2] = -2
-2.3	[-2.3] = -3

$\lfloor 2.4 \rfloor = 2,\quad \lfloor 1.9 \rfloor = 1,\quad \lfloor 0 \rfloor = 0,\quad \lfloor -1.2 \rfloor = -2,$

$\lfloor 2 \rfloor = 2,\quad \lfloor 0.2 \rfloor = 0,\quad \lfloor -0.3 \rfloor = -1\quad \lfloor -2 \rfloor = -2.$



Solved questions

Find the domain and range of each function.

1. $f(x) = 1 + x^2$ **2.** $f(x) = 1 - \sqrt{x}$

3. $F(t) = \dfrac{1}{\sqrt{t}}$ **4.** $F(t) = \dfrac{1}{1 + \sqrt{t}}$

5. $g(z) = \sqrt{4 - z^2}$ **6.** $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

Solution

1. domain $= (-\infty, \infty)$; range $= [1, \infty)$
2. domain $= [0, \infty)$; range $= (-\infty, 1]$
3. domain $= (0, \infty)$; y in range $\Rightarrow$ $y = \frac{1}{\sqrt{t}}$, $t > 0$ $\Rightarrow$ $y^2 = \frac{1}{t}$ and $y > 0$ $\Rightarrow$ y can be any positive real number $\Rightarrow$ range $= (0, \infty)$.
4. domain $= [0, \infty)$; y in range $\Rightarrow$ $y = \frac{1}{1+\sqrt{t}}$, $t > 0$. If $t = 0$, then $y = 1$ and as t increases, y becomes a smaller and smaller positive real number $\Rightarrow$ range $= (0, 1]$.
5. $4 - z^2 = (2 - z)(2 + z) \geq 0 \Leftrightarrow z \in [-2, 2] =$ domain. Largest value is $g(0) = \sqrt{4} = 2$ and smallest value is $g(-2) = g(2) = \sqrt{0} = 0 \Rightarrow$ range $= [0, 2]$.

❖ Find the domain and graph the functions in Exercises 15–20

15. $f(x) = 5 - 2x$ **16.** $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$ **18.** $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$ **20.** $G(t) = 1/|t|$

21. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$ **b.** $y^2 = x^2$

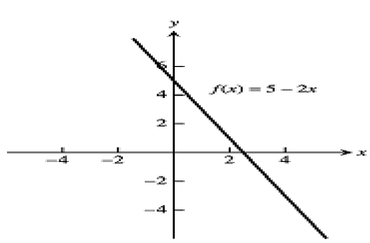
22. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

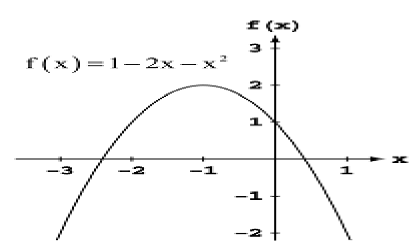
❖ Solution

We have first draw the function and then determine the domain

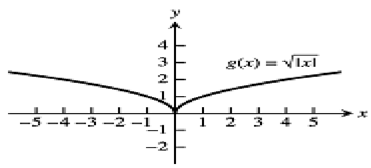
15. The domain is $(-\infty, \infty)$.



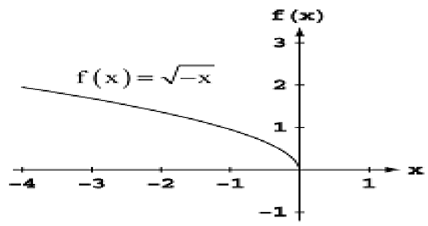
16. The domain is $(-\infty, \infty)$.



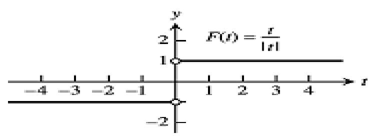
17. The domain is $(-\infty, \infty)$.



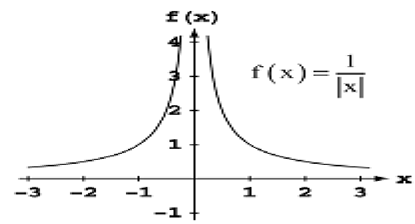
18. The domain is $(-\infty, 0]$.



19. The domain is $(-\infty, 0) \cup (0, \infty)$.



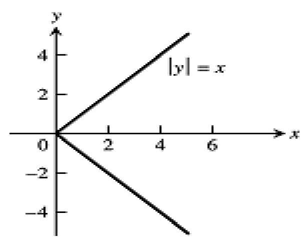
20. The domain is $(-\infty, 0) \cup (0, \infty)$.



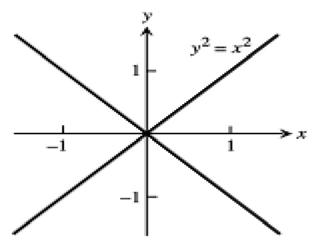
$$|x+y| = 1 \Leftrightarrow \left\{ \begin{array}{c} x+y=1 \\ \text{or} \\ x+y=-1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} y=1-x \\ \text{or} \\ y=-1-x \end{array} \right\}$$

21. Neither graph passes the vertical line test

(a)

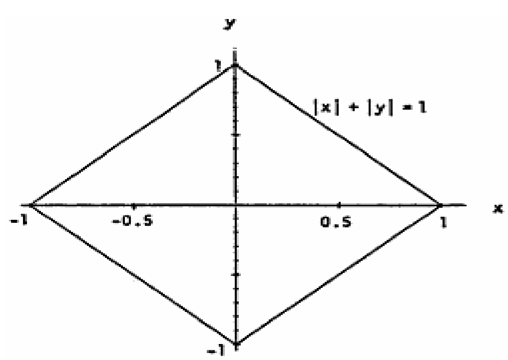


(b)

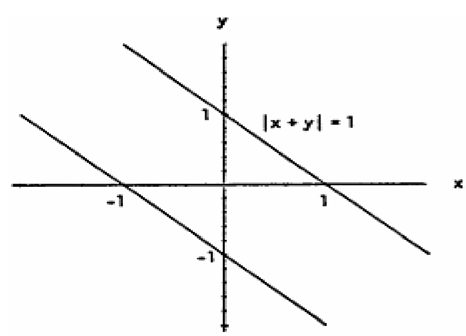


22. Neither graph passes the vertical line test

(a)



(b)



❖ *Greatest integer function Piecewise-Defined Functions*
Graph the functions in Exercises 23–26

23. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

24. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

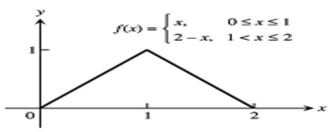
25. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

26. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

Solution

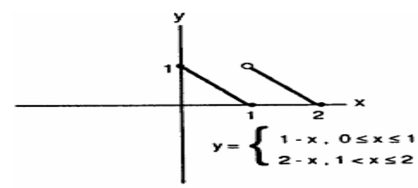
23.

x	0	1	2
y	0	1	0

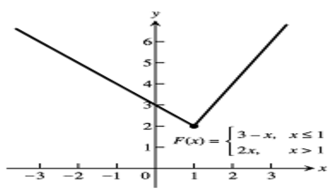


24.

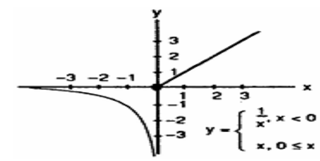
x	0	1	2
y	1	0	0



25. $y = \begin{cases} 3 - x, & x \le 1 \\ 2x, & 1 < x \end{cases}$

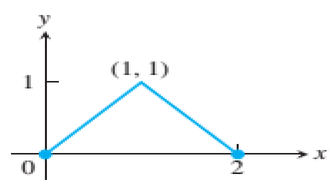


26. $y = \begin{cases} \frac{1}{x}, & x < 0 \\ x, & 0 \le x \end{cases}$

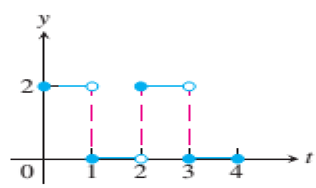


27. Find a formula for each function graphed

a.



b.



Solution

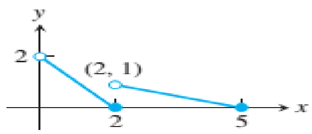
27. (a) Line through (0, 0) and (1, 1): $y = x$

Line through (1, 1) and (2, 0): $y = -x + 2$

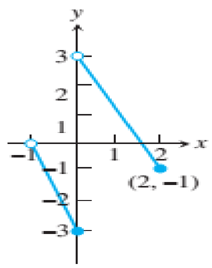
$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$$

(b) $$f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 0, & 3 \le x \le 4 \end{cases}$$

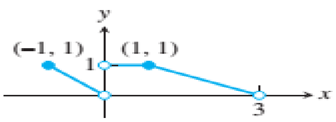
28. **a.**



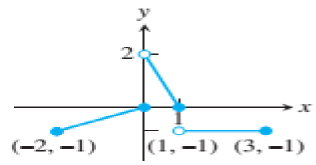
b.



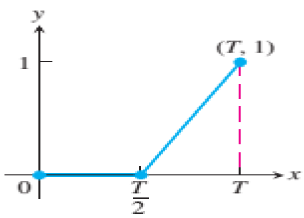
29. **a.**



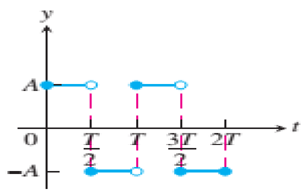
b.



30. **a.**



b.



31. **a.** Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

32. **a.** Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b. Confirm your findings in part (a) algebraically.

❖ Solution

31. (a) From the graph, $\frac{x}{2} > 1 + \frac{4}{x} \Rightarrow x \in (-2, 0) \cup (4, \infty)$

(b) $\frac{x}{2} > 1 + \frac{4}{x} \Rightarrow \frac{x}{2} - 1 - \frac{4}{x} > 0$

$x > 0$: $\frac{x}{2} - 1 - \frac{4}{x} > 0 \Rightarrow \frac{x^2-2x-8}{2x} > 0 \Rightarrow \frac{(x-4)(x+2)}{2x} > 0$

$\Rightarrow x > 4$ since x is positive;

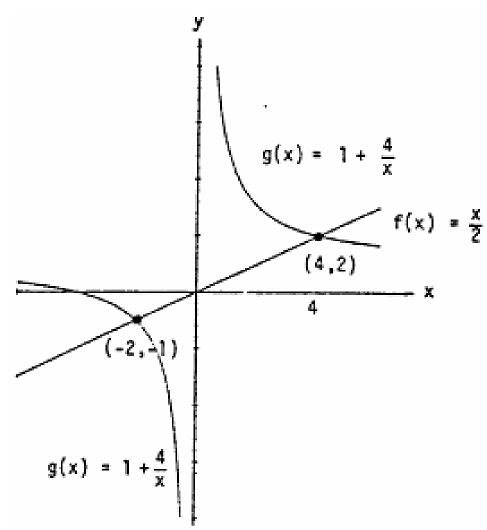
$x < 0$: $\frac{x}{2} - 1 - \frac{4}{x} > 0 \Rightarrow \frac{x^2-2x-8}{2x} < 0 \Rightarrow \frac{(x-4)(x+2)}{2x} < 0$

$\Rightarrow x < -2$ since x is negative;

sign of $(x - 4)(x + 2)$

$$\begin{array}{ccccc} + & & - & & + \\ & -2 & & 4 & \end{array}$$

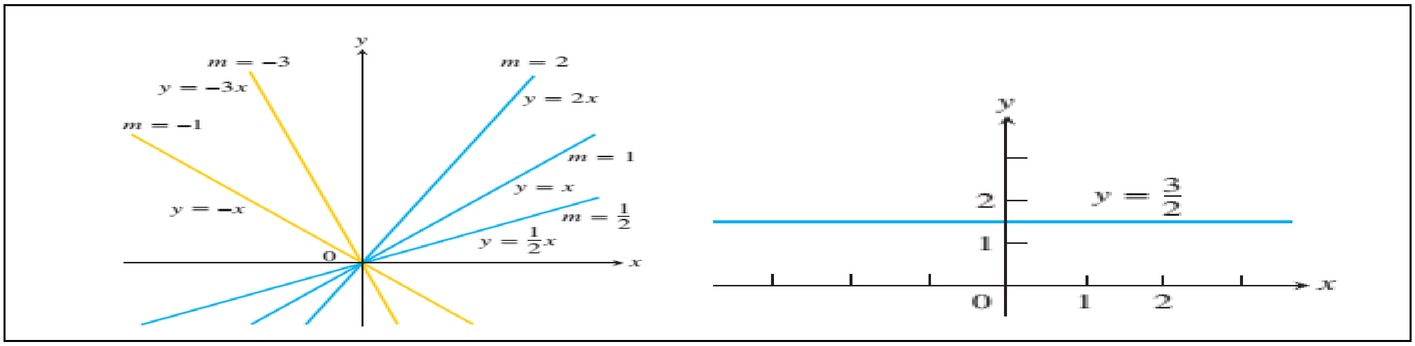
Solution interval: $(-2, 0) \cup (4, \infty)$



1.4 Identifying Functions; Mathematical Models

- There are a number of important types of functions frequently encountered in calculus. We identify and briefly summarize them here.

- **Linear Functions:** A function of the form for constants**(m and b)**, is called a **linear function**. Figure below shows an array of lines where so these lines pass through the **origin**. Constant functions result when the slope m=0 .



- **Power Functions**: A function **(f(x) = x^a)** where **a** is a constant, is called a **power function**. There are several important cases to consider.

(a) **a = n** ,,, a positive integer

The graphs of **(f(x) = x^n)** for n= 1, 2, 3, 4, 5,

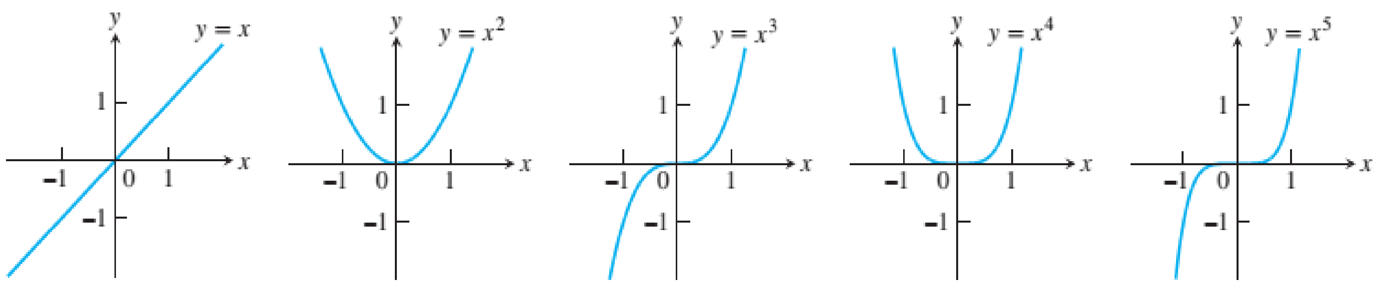


FIGURE 1.36 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

(b) $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in

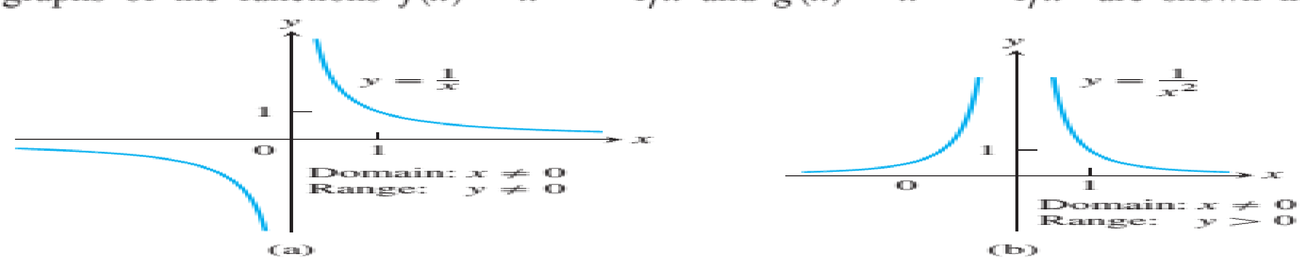


FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$

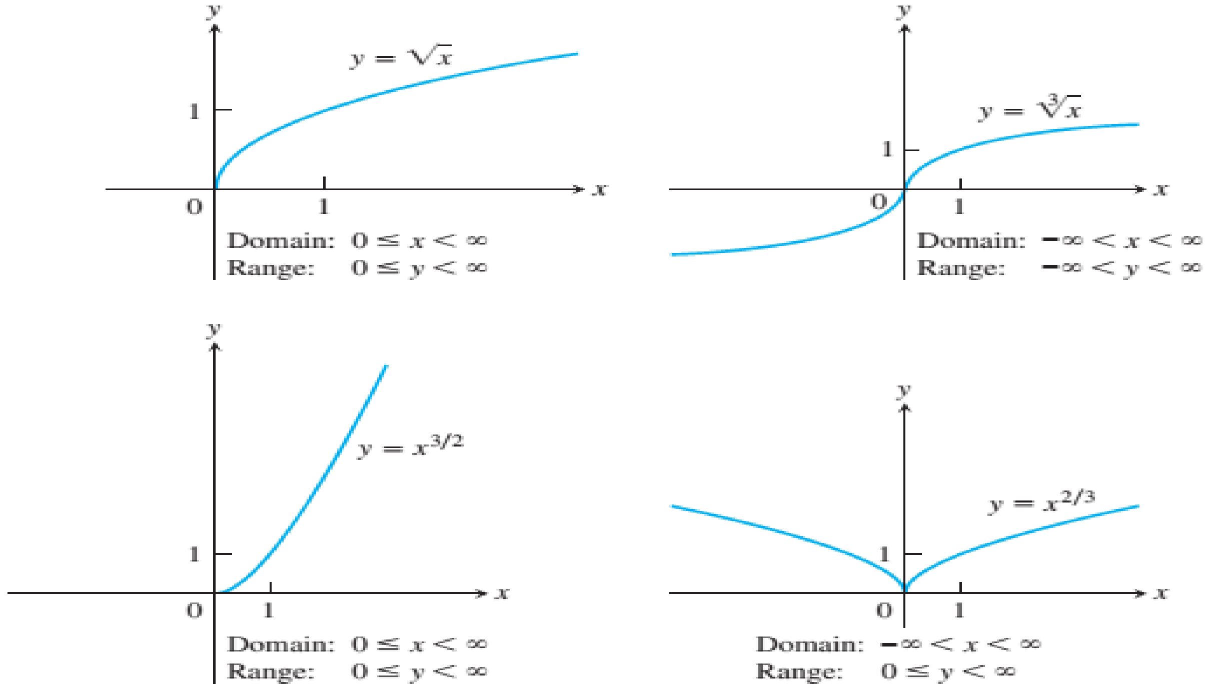


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

❖ **Polynomials**: A function **p** is **a polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where **(n)** is a nonnegative integer and the numbers **($a_0, a_1, a_2, ,\ldots\ldots a_n$)** are real constants (**called the coefficients of the polynomial**). All polynomials have domain **($-\infty, \infty$) .** If the leading coefficient **($a_n \neq 0$)** and **(n > 0)** then n is called the degree of the polynomial. Linear functions with ($m \neq 0$) are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$ are called **quadratic functions**. Likewise, cubic functions are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure below shows the graphs of three polynomials.

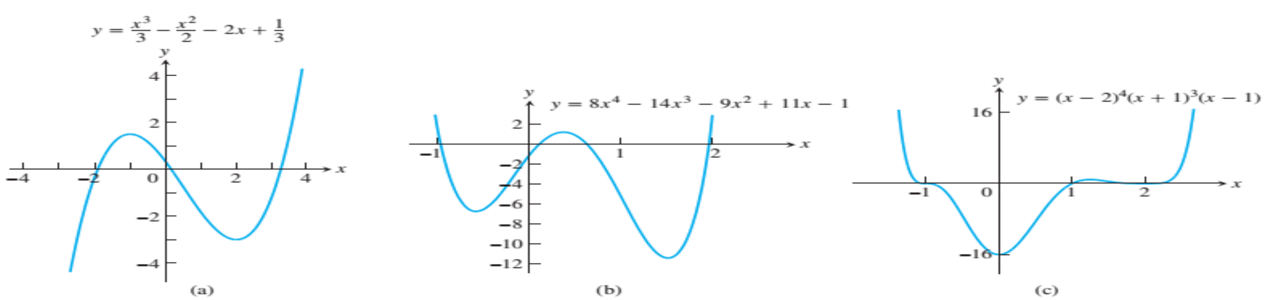


FIGURE 1.39 Graphs of three polynomial functions.

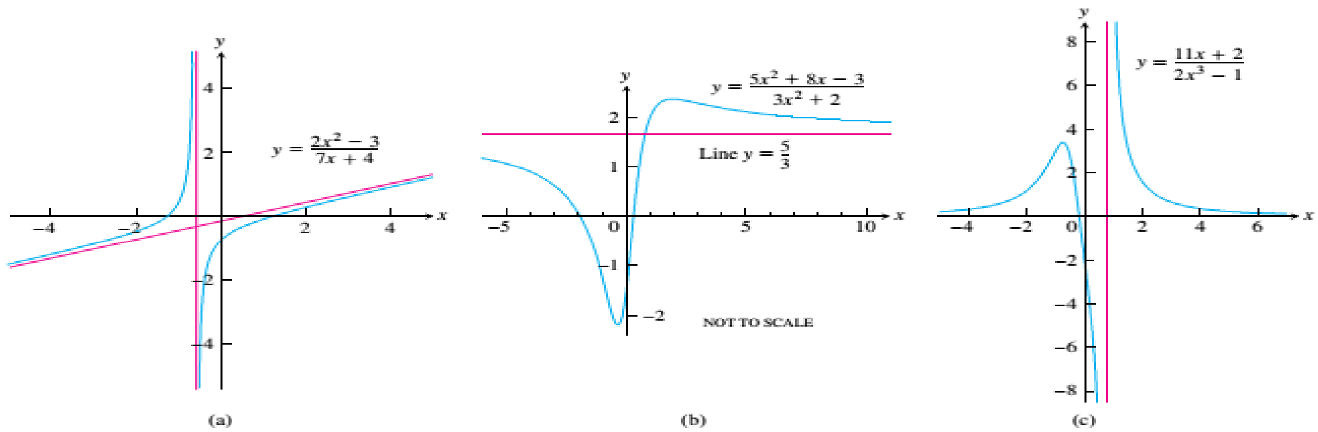
- **Rational Functions**: A rational function is a ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where **p** and **q** are polynomials. The domain of a rational function is the set of all real x for which **(q(x) ≠ 0)** For example,

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

the function is a rational function with domain $\{x \mid x \neq -4/7\}$ Its graph is shown below



- **Algebraic Functions:** An algebraic functionis a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots).
- **Exponential Functions**: Functions of the form **($f(x) = a^x$)** where($a>0$) the base is a positive constant **($a \neq 1$)** and are called **exponential functions**. All exponential functions have domain **($-\infty$, ∞)and range (0. ∞)** So an exponential function never assumes the value 0.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions,

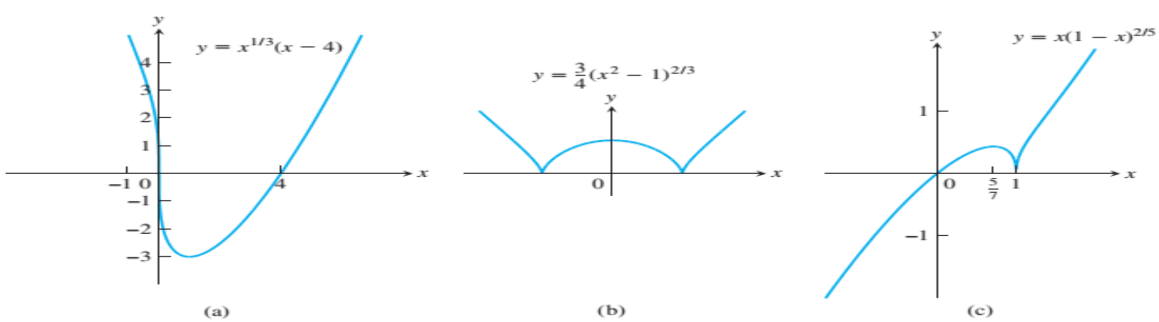


FIGURE 1.41 Graphs of three algebraic functions.

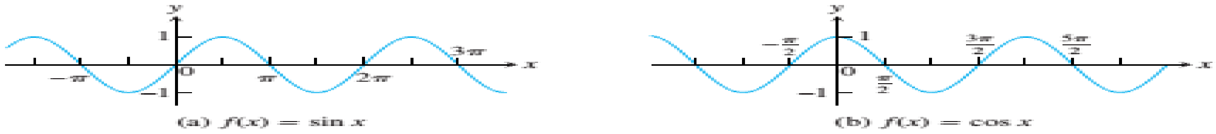


FIGURE 1.42 Graphs of the sine and cosine functions.

FIGURE 1.43 Graphs of exponential functions.

EXAMPLE 1 Recognizing Functions

Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category. For example, $f(x) = x^2$ is both a power function and a polynomial of second degree.

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ **(b)** $g(x) = 7^x$ **(c)** $h(z) = z^7$

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$

Solution

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ is a polynomial of degree 5.

(b) $g(x) = 7^x$ is an exponential function with base 7. Notice that the variable x is the exponent.

(c) $h(z) = z^7$ is a power function. (The variable z is the base.)

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$ is a trigonometric function. ■

Increasing Versus Decreasing Functions:

Function	Where increasing	Where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \le x < \infty$	Nowhere
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

Even Functions and Odd Functions: Symmetry

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$).

(i) For $f(x) = 3x^3 - x$ we have

$$f(-x) = 3(-x)^3 - (-x) = -3x^3 + x$$
$$= -(3x^3 - x) = -f(x)$$

So $3x^3 - x$ is an odd function.

(ii) For $f(x) = \dfrac{x^2}{1+x^2}$

$$f(-x) = \frac{(-x)^2}{1+(-x)^2} = \frac{x^2}{1+x^2} = f(x)$$

so this is even.

(iii) If $f(x) = \dfrac{2x}{x^2-1}$ then

$$f(-x) = \frac{2(-x)}{(-x)^2-1} = -\frac{2x}{x^2-1} = -f(x)$$

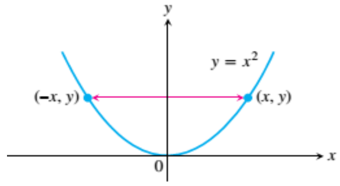
so $f(x)$ is odd.

(iv) If $f(x) = \dfrac{x^2}{x+1}$ we have

$$f(-x) = \frac{(-x)^2}{-x+1} = \frac{x^2}{1-x}$$

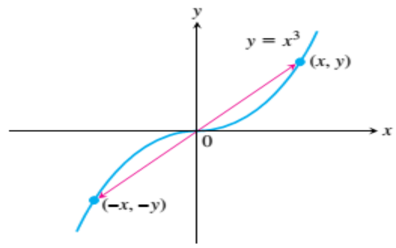
This is not equal to $f(x)$ or $-f(x)$ and so this function is neither odd nor even.

❖ The graph of an even function is **symmetric about the *y*-axis**. Since f(-x) = f(x)a point (x, y) lies on the graph if and only if the point (x,y) lies on the graph (Figure a). A reflection across the y-axis leaves the graph unchanged.



(a)

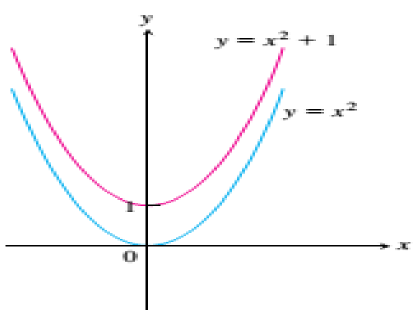
- ❖ The graph of an odd function is **symmetric about the origin**. Since f(-x) = -f(x) a point (*x*, *y*) lies on the graph if and only if the point(-*x*, -*y*) lies on the graph (Figure b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply both *x* and must be in the domain of *f*.



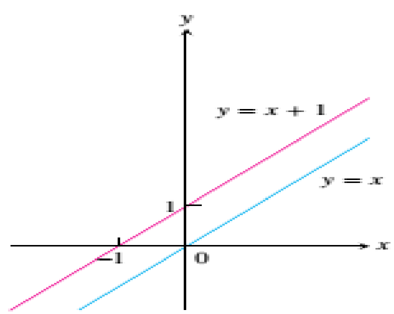
(b)

EXAMPLE 2 Recognizing Even and Odd Functions

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.47a).



(a)



(b)

FIGURE 1.47

$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.47b). ■

❖ Solved question

In Exercises 1–4, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

1. a. $f(x) = 7 - 3x$ **b.** $g(x) = \sqrt[5]{x}$

1. (a) linear, polynomial of degree 1, algebraic. (b) power, algebraic.

c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ **d.** $r(x) = 8^x$

(c) rational, algebraic. (d) exponential.

2. a. $F(t) = t^4 - t$ **b.** $G(t) = 5^t$

2. (a) polynomial of degree 4, algebraic. (b) exponential.

c. $H(z) = \sqrt{z^3 + 1}$ **d.** $R(z) = \sqrt[3]{z^7}$

(c) algebraic. (d) power, algebraic.

3. a. $y = \dfrac{3 + 2x}{x - 1}$ **b.** $y = x^{5/2} - 2x + 1$

c. $y = \tan \pi x$ **d.** $y = \log_7 x$

4. a. $y = \log_5 \left(\dfrac{1}{t}\right)$ **b.** $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$

c. $g(x) = 2^{1/x}$ **d.** $w = 5 \cos \left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$

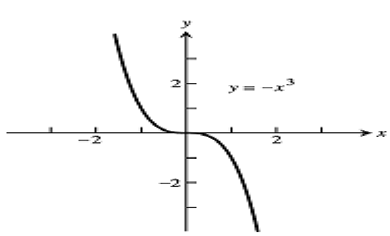
❖ Solved questions

Increasing and Decreasing Functions

Graph the functions in Exercises 7–18. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

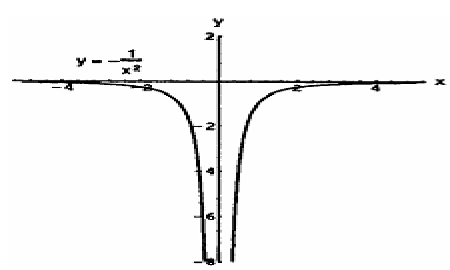
7. $y = -x^3$ **8.** $y = -\dfrac{1}{x^2}$

7. Symmetric about the origin
Dec: $-\infty < x < \infty$
Inc: nowhere



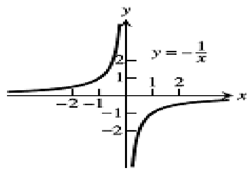
Dec is decrease
Inc is increase

8. Symmetric about the y-axis
Dec: $-\infty < x < 0$
Inc: $0 < x < \infty$

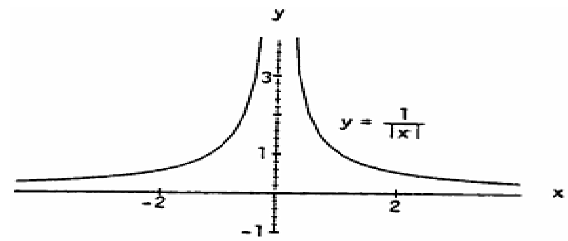


9. $y = -\dfrac{1}{x}$ **10.** $y = \dfrac{1}{|x|}$

9. Symmetric about the origin
Dec: nowhere
Inc: $-\infty < x < 0$
$0 < x < \infty$



10. Symmetric about the y-axis
Dec: $0 < x < \infty$
Inc: $-\infty < x < 0$



11. $y = \sqrt{|x|}$ **12.** $y = \sqrt{-x}$
13. $y = x^3/8$ **14.** $y = -4\sqrt{x}$
15. $y = -x^{3/2}$ **16.** $y = (-x)^{3/2}$
17. $y = (-x)^{2/3}$ **18.** $y = -x^{2/3}$

❖ Solved question

Even and Odd Functions

In Exercises 19–30, say whether the function is even, odd, or neither. Give reasons for your answer

19. $f(x) = 3$ **20.** $f(x) = x^{-5}$

19. Since a horizontal line not through the origin is symmetric with respect to the y-axis, but not with respect to the origin, the function is even.

20. $f(x) = x^{-5} = \frac{1}{x^5}$ and $f(-x) = (-x)^{-5} = \frac{1}{(-x)^5} = -\left(\frac{1}{x^5}\right) = -f(x)$. Thus the function is odd.

21. $f(x) = x^2 + 1$ **22.** $f(x) = x^2 + x$

21. Since $f(x) = x^2 + 1 = (-x)^2 + 1 = -f(x)$. The function is even.

22. Since $[f(x) = x^2 + x] \neq [f(-x) = (-x)^2 - x]$ and $[f(x) = x^2 + x] \neq [-f(x) = -(x)^2 - x]$ the function is neither even nor odd.

23. $g(x) = x^3 + x$ **24.** $g(x) = x^4 + 3x^2 - 1$

23. Since $g(x) = x^3 + x$, $g(-x) = -x^3 - x = -(x^3 + x) = -g(x)$. So the function is odd.

24. $g(x) = x^4 + 3x^2 + 1 = (-x)^4 + 3(-x)^2 - 1 = g(-x)$, thus the function is even.

25. $g(x) = \dfrac{1}{x^2 - 1}$ **26.** $g(x) = \dfrac{x}{x^2 - 1}$

27. $h(t) = \dfrac{1}{t - 1}$ **28.** $h(t) = |t^3|$

29. $h(t) = 2t + 1$ **30.** $h(t) = 2|t| + 1$

1.5 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$) (that is, for), we define functions and fg by the formulas

$$(f+g)(x) = f(x) + g(x).$$
$$(f-g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 Combining Functions Algebraically

The functions defined by the formulas

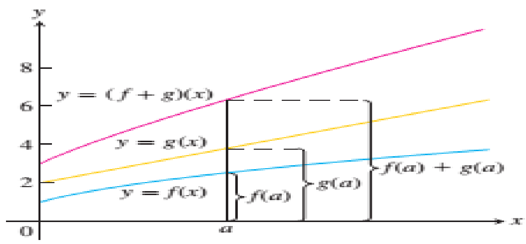
$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1-x},$$

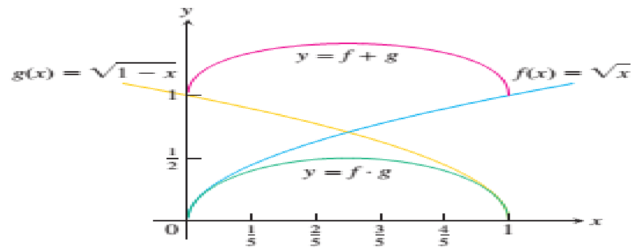
have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

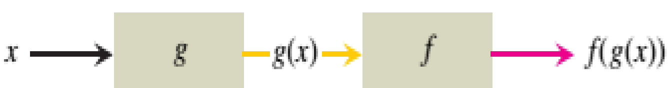


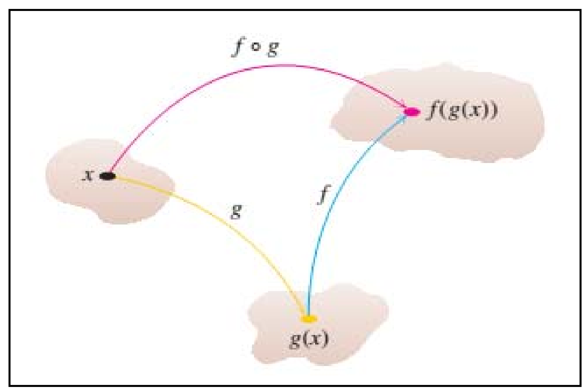


- **<u>Composite Functions</u>**
- Composition is another method for combining functions.

> **DEFINITION Composition of Functions**
> If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by
> $$(f \circ g)(x) = f(g(x)).$$
> The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

- The definition says that **(f0g)** can be formed when the range of **(g)** lies in the domain of ***f***. To find (**(f0g)(x)**) first **find g(x)** and second find ***f*(g(x)).**





EXAMPLE 3 Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

- ❖ **<u>Shifting a Graph of a Function</u>**
- ❖ To shift the graph of a function **(y= f(x))** straight up, add a positive constant to the right hand side of the formula **(y= f(x)).**
- ❖ To shift the graph of a function **(y= f(x))** straight down, add a negative constant to the right-hand side of the formula **(y= f(x)).**
- ❖ To shift the graph of **(y= f(x))** to the left, add a positive constant to x.
- ❖ To shift the graph of **(y= f(x))** to the right, add a negative constant to x.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k\rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h\rvert$ units if $h < 0$

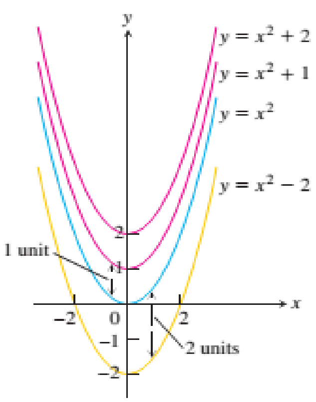


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

EXAMPLE 4 Shifting a Graph

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.54).

(b) Adding −2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.54).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.55).

(d) Adding −2 to x in $y = |x|$, and then adding −1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.56).

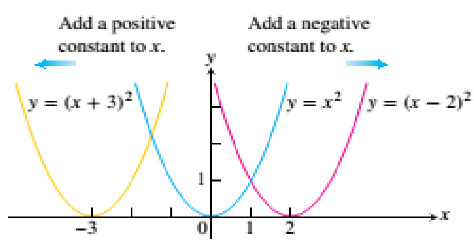


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

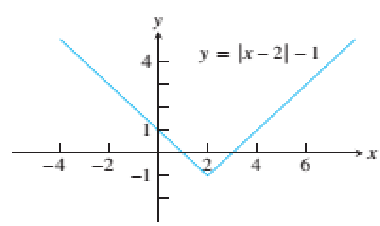


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d). ■

- ❖ Solved question

Sums, Differences, Products, and Quotients

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$

2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

1. $D_f: -\infty < x < \infty, D_g: x \geq 1 \Rightarrow D_{f+g} = D_{fg}: x \geq 1.$ $R_f: -\infty < y < \infty, R_g: y \geq 0, R_{f+g}: y \geq 1, R_{fg}: y \geq 0$

2. $D_f: x + 1 \geq 0 \Rightarrow x \geq -1, D_g: x - 1 \geq 0 \Rightarrow x \geq 1.$ Therefore $D_{f+g} = D_{fg}: x \geq 1.$
$R_f = R_g: y \geq 0, R_{f+g}: y \geq \sqrt{2}, R_{fg}: y \geq 0$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$

4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

3. $D_f: -\infty < x < \infty, D_g: -\infty < x < \infty \Rightarrow D_{f/g}: -\infty < x < \infty$ since $g(x) \neq 0$ for any x; $D_{g/f}: -\infty < x < \infty$ since $f(x) \neq 0$ for any x. $R_f: y = 2, R_g: y \geq 1, R_{f/g}: 0 < y \leq 2, R_{g/f}: y \geq \frac{1}{2}$

4. $D_f: -\infty < x < \infty, D_g: x \geq 0 \Rightarrow D_{f/g}: x \geq 0$ since $g(x) \neq 0$ for any $x \geq 0$; $D_{g/f}: x \geq 0$ since $f(x) \neq 0$ for any $x \geq 0$. $R_f: y = 1, R_g: y \geq 1, R_{f/g}: 0 < y \leq 1, R_{g/f}: y \geq 1$

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(-5))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

solution

5. (a) $f(g(0)) = f(-3) = 2$
(b) $g(f(0)) = g(5) = 22$
(c) $f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2$
(d) $g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 22$
(e) $f(f(-5)) = f(0) = 5$
(f) $g(g(2)) = g(1) = -2$
(g) $f(f(x)) = f(x + 5) = (x + 5) + 5 = x + 10$
(h) $g(g(x)) = g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a. $f(g(1/2))$ **b.** $g(f(1/2))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(2))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

7. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a. $u(v(f(x)))$ **b.** $u(f(v(x)))$
c. $v(u(f(x)))$ **d.** $v(f(u(x)))$
e. $f(u(v(x)))$ **f.** $f(v(u(x)))$

8. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a. $h(g(f(x)))$ **b.** $h(f(g(x)))$
c. $g(h(f(x)))$ **d.** $g(f(h(x)))$
e. $f(g(h(x)))$ **f.** $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 9 and 10 as a composite involving one or more of f, g, h, and j.

9. a. $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
c. $y = x^{1/4}$ **d.** $y = 4x$
e. $y = \sqrt{(x-3)^3}$ **f.** $y = (2x - 6)^3$

10. a. $y = 2x - 3$ **b.** $y = x^{3/2}$
c. $y = x^9$ **d.** $y = x - 6$
e. $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$

solution

9. (a) $y = f(g(x))$ (b) $y = j(g(x))$
(c) $y = g(g(x))$ (d) $y = j(j(x))$
(e) $y = g(h(f(x)))$ (f) $y = h(j(f(x)))$

10. (a) $y = f(j(x))$ (b) $y = h(g(x)) = g(h(x))$
(c) $y = h(h(x))$ (d) $y = f(f(x))$
(e) $y = j(g(f(x)))$ (f) $y = g(f(h(x)))$

11. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	
b.	$x + 2$	$3x$	
c.		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\frac{x}{x-1}$	$\frac{x}{x-1}$	
e.		$1 + \frac{1}{x}$	x
f.	$\frac{1}{x}$		x

12. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\frac{1}{x-1}$	$\|x\|$	?
b.	?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c.	?	$\sqrt{x}$	$\|x\|$
d.	$\sqrt{x}$	?	$\|x\|$

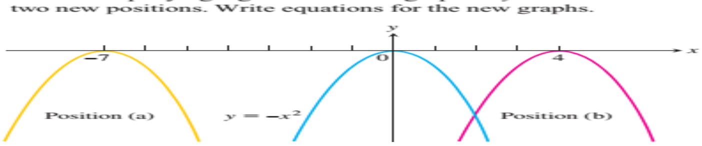
solution

11.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	$x-7$	$\sqrt{x}$	$\sqrt{x-7}$
(b)	$x+2$	$3x$	$3(x+2)=3x+6$
(c)	x^2	$\sqrt{x-5}$	$\sqrt{x^2-5}$
(d)	$\frac{x}{x-1}$	$\frac{x}{x-1}$	$\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=\frac{x}{x-(x-1)}=x$
(e)	$\frac{1}{x-1}$	$1+\frac{1}{x}$	x
(f)	$\frac{1}{x}$	$\frac{1}{x}$	x

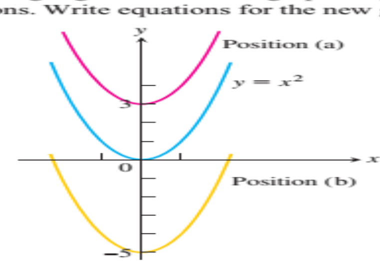
Shifting Graphs

15. The accompanying figure shows the graph of $y=-x^2$ shifted to two new positions. Write equations for the new graphs.



15. (a) $y=-(x+7)^2$ (b) $y=-(x-4)^2$

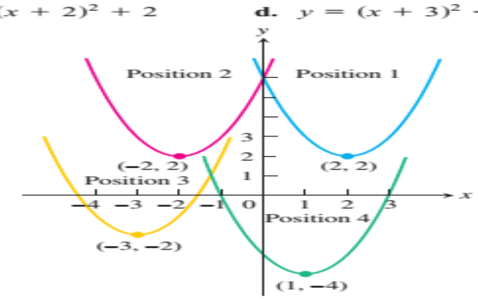
16. The accompanying figure shows the graph of $y=x^2$ shifted to two new positions. Write equations for the new graphs.



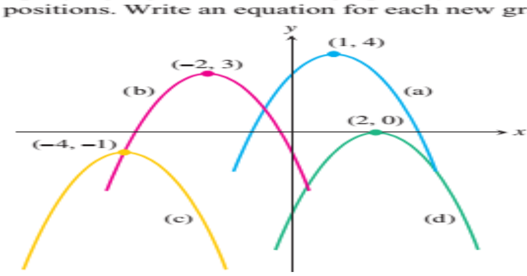
17. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y=(x-1)^2-4$ **b.** $y=(x-2)^2+2$

c. $y=(x+2)^2+2$ **d.** $y=(x+3)^2-2$



18. The accompanying figure shows the graph of $y=-x^2$ shifted to four new positions. Write an equation for each new graph.



16. (a) $y = x^2 + 3$ (b) $y = x^2 - 5$

17. (a) Position 4 (b) Position 1 (c) Position 2 (d) Position 3

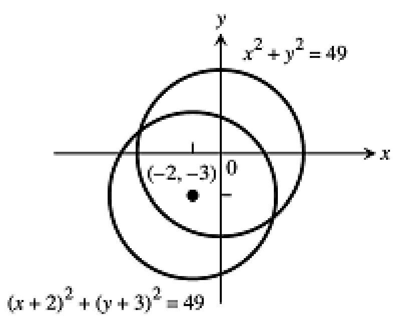
- Exercises 19–28 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

19. $x^2 + y^2 = 49$ Down 3, left 2

20. $x^2 + y^2 = 25$ Up 3, left 4

solution

19.



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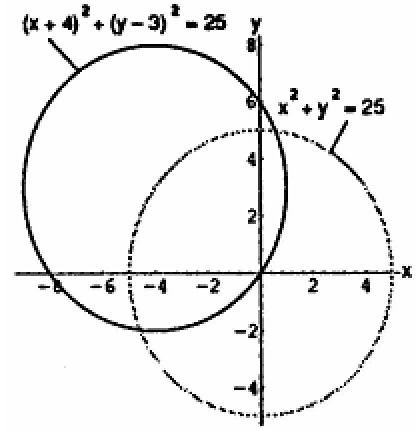
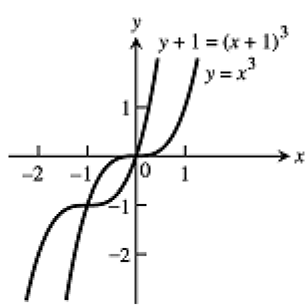


21. $y = x^3$ Left 1, down 1

22. $y = x^{2/3}$ Right 1, down 1

solution

21.



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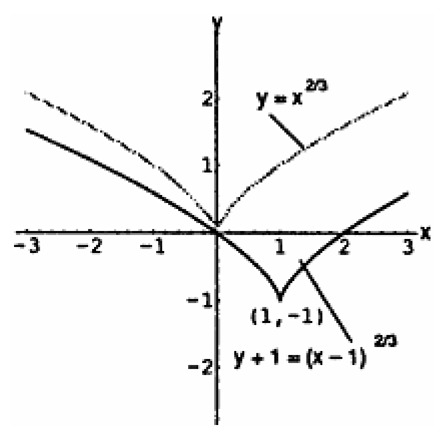


23. $y = \sqrt{x}$ Left 0.81

24. $y = -\sqrt{x}$ Right 3

25. $y = 2x - 7$ Up 7

26. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

27. $y = 1/x$ Up 1, right 1

28. $y = 1/x^2$ Left 2, down 1

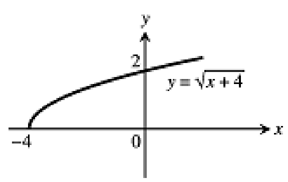
Graph the functions in Exercises 29–48.

29. $y = \sqrt{x + 4}$ **30.** $y = \sqrt{9 - x}$

31. $y = |x - 2|$ **32.** $y = |1 - x| - 1$

solution

29.



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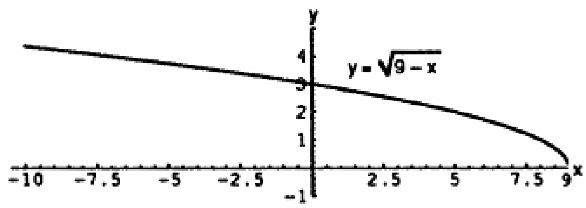

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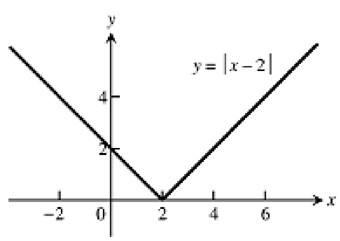
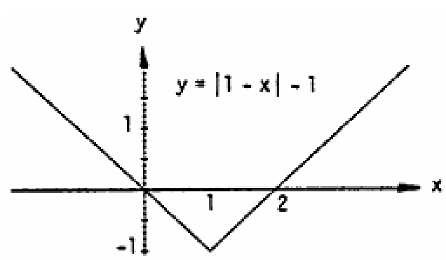


32.



33. $y = 1 + \sqrt{x - 1}$ **34.** $y = 1 - \sqrt{x}$

35. $y = (x + 1)^{2/3}$ **36.** $y = (x - 8)^{2/3}$

37. $y = 1 - x^{2/3}$ **38.** $y + 4 = x^{2/3}$

39. $y = \sqrt[3]{x - 1} - 1$ **40.** $y = (x + 2)^{3/2} + 1$

41. $y = \frac{1}{x - 2}$ **42.** $y = \frac{1}{x} - 2$

43. $y = \frac{1}{x} + 2$ **44.** $y = \frac{1}{x + 2}$

45. $y = \frac{1}{(x - 1)^2}$ **46.** $y = \frac{1}{x^2} - 1$

47. $y = \frac{1}{x^2} + 1$ **48.** $y = \frac{1}{(x + 1)^2}$

Chapter Three

Limits Using the Limit Laws

This section presents theorems for calculating limits.

The Limit Laws

This theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x\to c}(f(x)\cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x\to c}\dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a)

$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Product and Multiple Rules}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to -2}\sqrt{4x^2 - 3} &= \sqrt{\lim_{x\to -2}(4x^2 - 3)} && \text{Power Rule with } r/s = \tfrac{1}{2}\\ &= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} && \text{Difference Rule}\\ &= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules}\\ &= \sqrt{16 - 3}\\ &= \sqrt{13}\end{aligned}$$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x\to -1}\frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step.

- Theorem 3 applies only if the denominator of the rational function is not zero at the limit point *c*. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at *c*. If this happens, we can find the limit by substitution in the simplified fraction. (**Eliminating Zero Denominators Algebraically**)
- Identifying Common Factors It can be shown that if **Q(x)** is a polynomial and **Q(c) =0,** then $(x-c)$ is a factor of Q(x). Thus, if the numerator and denominator of a rational function of x are both zero at $x=c$ they have $(x-c)$ as a common factor.

EXAMPLE 3 Canceling a Common Factor

Evaluate

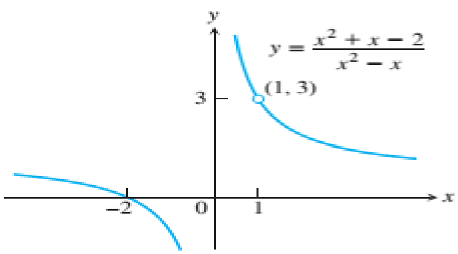
$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x-1)$ in common with the denominator. Canceling the $(x-1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

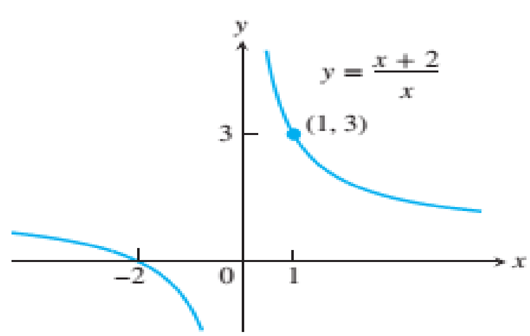
$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$



(a)



(b)

EXAMPLE 4 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

Solution This is the limit we considered in Example 10 of the preceding section. We cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors. We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}\\
&= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}\\
&= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} && \text{Common factor } x^2\\
&= \frac{1}{\sqrt{x^2+100}+10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}\\
&= \frac{1}{\sqrt{0^2+100}+10} && \text{Denominator not 0 at } x=0; \text{ substitute}\\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

- ❖ Solved question
- ❖ Find the limits in Exercises 1–18

1. $\lim_{x\to -7}(2x+5)$ **2.** $\lim_{x\to 12}(10-3x)$

solution

1. $\lim_{x\to -7}(2x+5) = 2(-7)+5 = -14+5 = -9$ 2. $\lim_{x\to 12}(10-3x) = 10-3(12) = 10-36 = -26$

3. $\lim_{x\to 2}(-x^2+5x-2)$ **4.** $\lim_{x\to -2}(x^3-2x^2+4x+8)$

solution

3. $\lim_{x\to 2}(-x^2+5x-2) = -(2)^2+5(2)-2 = -4+10-2 = 4$

4. $\lim_{x\to -2}(x^3-2x^2+4x+8) = (-2)^3-2(-2)^2+4(-2)+8 = -8-8-8+8 = -16$

5. $\lim_{t\to 6} 8(t-5)(t-7)$

6. $\lim_{s\to 2/3} 3s(2s-1)$

7. $\lim_{x\to 2} \frac{x+3}{x+6}$

8. $\lim_{x\to 5} \frac{4}{x-7}$

9. $\lim_{y\to -5} \frac{y^2}{5-y}$

10. $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$

11. $\lim_{x\to -1} 3(2x-1)^2$

12. $\lim_{x\to -4} (x+3)^{1984}$

13. $\lim_{y\to -3} (5-y)^{4/3}$

14. $\lim_{z\to 0} (2z-8)^{1/3}$

15. $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$

16. $\lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2}$

17. $\lim_{h\to 0} \frac{\sqrt{3h+1}-1}{h}$

18. $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$

❖ Find the limits in Exercises 19–36.

19. $\lim_{x\to 5} \frac{x-5}{x^2-25}$

20. $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$

solution

19. $$\lim_{x\to 5} \frac{x-5}{x^2-25} = \lim_{x\to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x\to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$$

20. $$\lim_{x\to -3} \frac{x+3}{x^2+4x+3} = \lim_{x\to -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x\to -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$$

21. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$

22. $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

solution

21. $$\lim_{x\to -5} \frac{x^2+3x-10}{x+5} = \lim_{x\to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x\to -5} (x-2) = -5-2 = -7$$

22. $$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \lim_{x\to 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x\to 2} (x-5) = 2-5 = -3$$

23. $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$

24. $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

25. $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2}$

26. $\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2}$

27. $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

28. $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

29. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

30. $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$

31. $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

32. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

33. $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

34. $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

35. $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$ **36.** $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

2.5 Infinite Limits

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty .$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty .$$

If $f(x) = \frac{1}{x}$

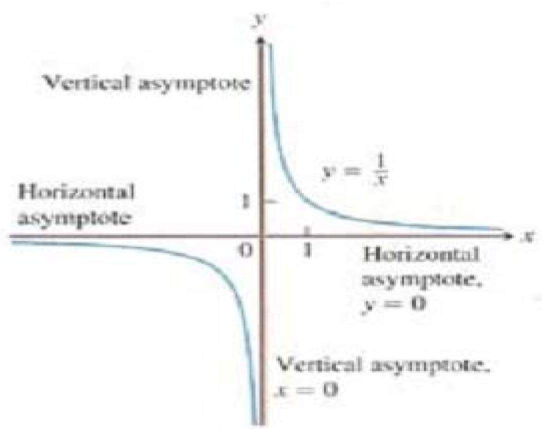
It could be seen from the graph:

1- as x approaches 0 from right then 1/x tends to $+\infty$

2- as x approached 0 from the left then 1/x tends to $-\infty$

3- as x approached ∞ from the right $(+\infty)$, then 1/x tends to 0

4- as x approached ∞ from the right $(-\infty)$, then 1/x tends to 0



so

$$\lim_{x\to +\infty} \frac{1}{x} = 0 \quad , \quad \lim_{x\to -\infty} \frac{1}{x} = 0$$

And $$\lim_{x\to 0^+} \frac{1}{x} = +\infty \quad , \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Example

$$\lim_{x\to\infty}(5+\frac{1}{x}) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5$$

Example

$$\lim_{x\to -\infty} \frac{\pi\sqrt{3}}{x^2} = \pi\sqrt{3} * 0 * 0 = 0$$

Example

Find $\lim_{x\to\infty} \sin(\frac{1}{x})$

Solution

Let $t = 1/x$

So $t \longrightarrow 0^+$ as $x \longrightarrow \infty$

So $\lim_{x\to\infty} \sin(\frac{1}{x}) = 0$

Vertical and Horizontal asymptotes

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty .$$

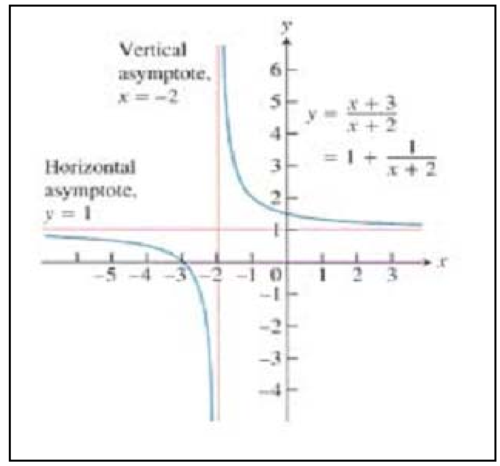
Defintion

- A line y = b is a horizontal asymptote of the graph function y= f(x) if either;

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

- A line x = a is a vertical asymptote of the graph of the function y = f(x) if

either; $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$



Example

$$\lim_{x\to\infty} \frac{x+3}{x+2}$$

Solution

$$\lim_{x\to\infty} \frac{1+3/x}{1+2/x} = 1$$ i.e y = 1 (horizontal asymptote)

$$\lim_{x\to -2} \frac{x+3}{x+2} = \infty$$ i.e. x = -2 (vertical asymptote)

EXAMPLE 3 Rational Functions Can Behave in Various Ways Near Zeros of Their Denominators

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ The values are negative for $x > 2$, x near 2.

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ The values are positive for $x < 2$, x near 2.

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. See parts (c) and (d).

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero in the denominator.

EXAMPLE 4 Using the Definition of Infinite Limits

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if } x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

EXAMPLE 5 Looking for Asymptotes

Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$.

$$\begin{array}{r} 1 \\ x + 2 \overline{\smash{)}\, x + 3} \\ \underline{x + 2} \\ 1 \end{array}$$

This result enables us to rewrite y:

$$y = 1 + \frac{1}{x + 2}.$$

We now see that the curve in question is the graph of $y = 1/x$ shifted 1 unit up and 2 units left (Figure 2.43). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$.

EXAMPLE 7 Curves with Infinitely Many Asymptotes

The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.45).

FIGURE 2.45 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 7).

The graphs of

$$y = \csc x = \frac{1}{\sin x} \qquad \text{and} \qquad y = \cot x = \frac{\cos x}{\sin x}$$

have vertical asymptotes at integer multiples of π, where $\sin x = 0$ (Figure 2.46).

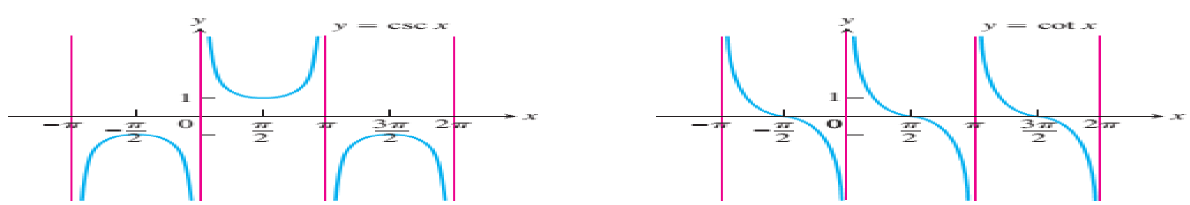


FIGURE 2.46 The graphs of $\csc x$ and $\cot x$ (Example 7).

Solved question

❖ Find the limits in Exercises 17–22

17. $\lim \dfrac{1}{x^2-4}$ as

a. $x \to 2^+$ **b.** $x \to 2^-$

c. $x \to -2^+$ **d.** $x \to -2^-$

Solution

17. (a) $\lim\limits_{x \to 2^+} \frac{1}{x^2-4} = \lim\limits_{x \to 2^+} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x \to 2^-} \frac{1}{x^2-4} = \lim\limits_{x \to 2^-} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x \to -2^+} \frac{1}{x^2-4} = \lim\limits_{x \to -2^+} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x \to -2^-} \frac{1}{x^2-4} = \lim\limits_{x \to -2^-} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{negative}\cdot\text{negative}}\right)$

18. $\lim \dfrac{x}{x^2-1}$ as

a. $x \to 1^+$ **b.** $x \to 1^-$

c. $x \to -1^+$ **d.** $x \to -1^-$

Solution

18. (a) $\lim\limits_{x \to 1^+} \frac{x}{x^2-1} = \lim\limits_{x \to 1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x \to 1^-} \frac{x}{x^2-1} = \lim\limits_{x \to 1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x \to -1^+} \frac{x}{x^2-1} = \lim\limits_{x \to -1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x \to -1^-} \frac{x}{x^2-1} = \lim\limits_{x \to -1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{negative}}\right)$

19. $\lim \left(\dfrac{x^2}{2} - \dfrac{1}{x}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to \sqrt[3]{2}$ **d.** $x \to -1$

20. $\lim \dfrac{x^2-1}{2x+4}$ as

a. $x \to -2^+$ **b.** $x \to -2^-$

c. $x \to 1^+$ **d.** $x \to 0^-$

21. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \to 0^+$ **b.** $x \to 2^+$

c. $x \to 2^-$ **d.** $x \to 2$

e. What, if anything, can be said about the limit as $x \to 0$?

22. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \to 2^+$ **b.** $x \to -2^+$

c. $x \to 0^-$ **d.** $x \to 1^+$

e. What, if anything, can be said about the limit as $x \to 0$?

❖ Find the limits in Exercises 23–26.

23. $\lim \left(2 - \dfrac{3}{t^{1/3}}\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

24. $\lim \left(\dfrac{1}{t^{3/5}} + 7\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

Solution

23. (a) $\lim\limits_{t \to 0^+} \left[2 - \frac{3}{t^{1/3}}\right] = -\infty$ (b) $\lim\limits_{t \to 0^-} \left[2 - \frac{3}{t^{1/3}}\right] = \infty$

24. (a) $\lim\limits_{t \to 0^+} \left[\frac{1}{t^{3/5}} + 7\right] = \infty$ (b) $\lim\limits_{t \to 0^-} \left[\frac{1}{t^{3/5}} + 7\right] = -\infty$

25. $\lim \left(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x-1)^{2/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

26. $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x-1)^{4/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

CHAPTER Four

2.6 Continuity

Any function whose graph can be sketched over its domain in one continuous motion without lifting the pencil is an example of a continuous function.

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

A function f(x) is continuous at x =c if and only if it meets the following three conditions:

1- f(c) exists (c lies in the domain of f)

2- $\lim_{x \to c} f(x)$ exists (f has a limit as x approaches c)

i.e $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c} f(x)$

3- $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

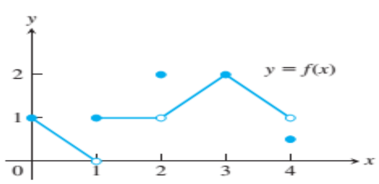


FIGURE 2.50 The function is continuous on [0, 4] except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\lim_{x \to 0^+} f(x) = f(0)$.

At $x = 3$, $\lim_{x \to 3} f(x) = f(3)$.

At $0 < c < 4$, $c \neq 1, 2$, $\lim_{x \to c} f(x) = f(c)$.

Points at which f is discontinuous:

At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $x = 4$, $\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.

At $c < 0$, $c > 4$, these points are not in the domain of f. ■

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 2.51).

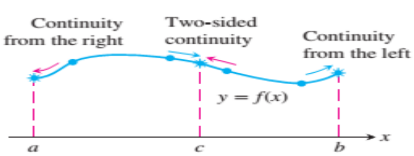
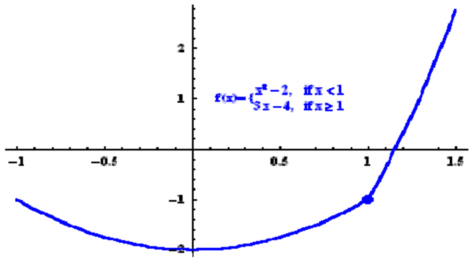


FIGURE 2.51 Continuity at points a, b, and c.

Exercise 1. Find *A* which makes the function continuous at x=1.

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x < 1 \\ Ax - 4 & \text{if } 1 \le x \end{cases}$$



We have

$$\lim_{x\to 1-} f(x) = \lim_{x \to 1-} x^2 - 2 = -1,$$

and

$$\lim_{x\to 1+} f(x) = \lim_{x \to 1+} Ax - 4 = A - 4.$$

So $f(x)$ is continuous at 1 iff

$A - 4 = -1$ or equivalently if $A = 3$.

Examples. at **x = 3**

$$f(x) = \frac{x^2 - 9}{x - 3}$$

fails to be continuous at **x = 3** since **3** is not in the domain of **f**.

Examples. at **x = 3**

$$h(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$ at **x = 3**

The function does satisfy condition 1 since 3 is in the domain of **h**, **h(3) = 7**, and does satisfy condition 2 since

$$h(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} x + 3 = 6$$

does exist. However, since these two numbers are different, condition 3 is violated and **h** fails to be continuous at **x = 3**.

Continuous Functions

THEOREM 9 **Properties of Continuous Functions**
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

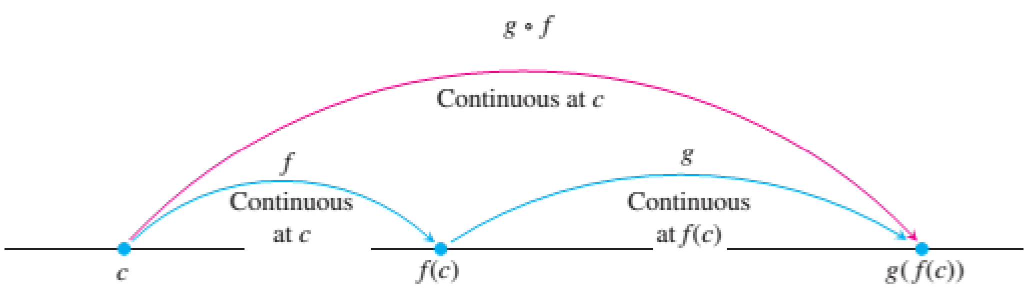
1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Most of the results in Theorem 9 are easily proved from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$
\begin{aligned}
\lim_{x \to c}(f + g)(x) &= \lim_{x \to c}(f(x) + g(x)) \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x), && \text{Sum Rule, Theorem 1} \\
&= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
&= (f + g)(c).
\end{aligned}
$$

This shows that $f + g$ is continuous.

THEOREM 10 **Composite of Continuous Functions**
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



EXAMPLE 8 Applying Theorems 9 and 10
Show that the following functions are continuous everywhere on their respective domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a rational power of the continuous identity function $f(x) = x$ (Part 6, Theorem 9). The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$.

(b) The numerator is a rational power of the identity function; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

(d) Because the sine function is everywhere-continuous (Exercise 62), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.58).

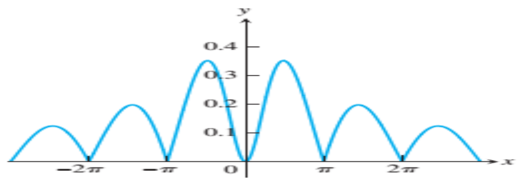


FIGURE 2.58 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

Continuous Extension to a Point

The function $y = (\sin x)/x$ is continuous at every point except $x = 0$

$y = (\sin x)/x$ is different from $y = 1/x$

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

The function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \to 0} \frac{\sin x}{x} = F(0)$$

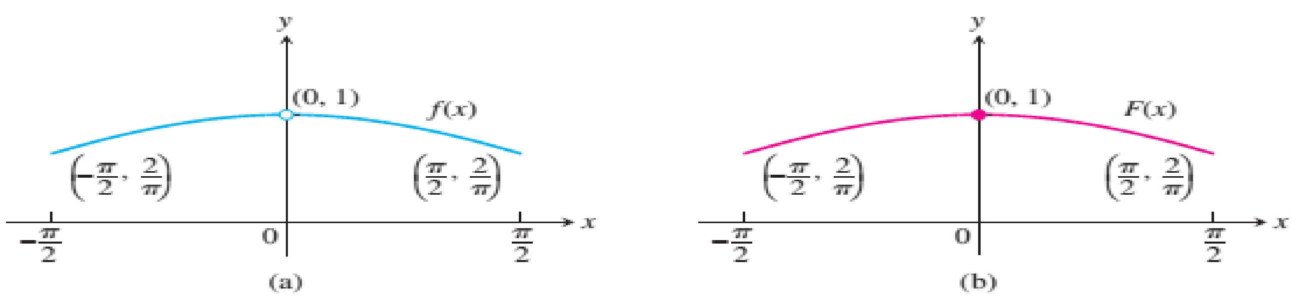


FIGURE 2.59 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$.

EXAMPLE 9 A Continuous Extension

Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The new function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of 5/4. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.60. The continuous extension F has the same graph except with no hole at (2, 5/4). Effectively, F is the function f with its point of discontinuity at $x = 2$ removed.

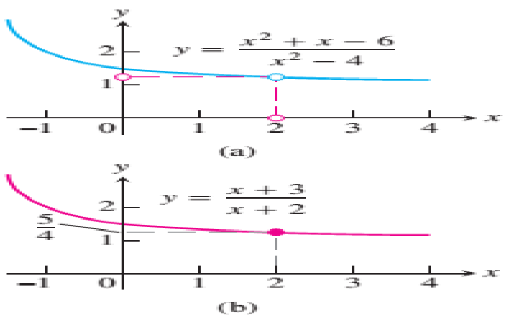
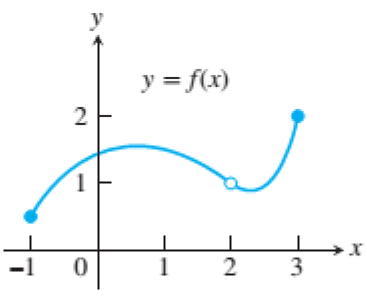


FIGURE 2.60 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 9).

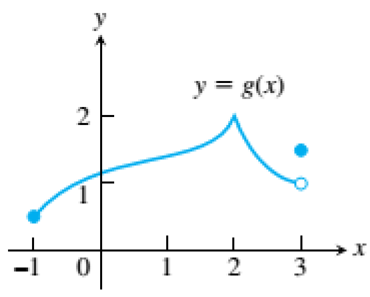
Solved question

In Exercises 1–2, say whether the function graphed is continuous on [-1,3] If not, where does it fail to be continuous and why?

1.



2.



Solution

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim_{x \to 3^-} g(x) \neq g(3) = 1.5$

At what points are the functions in Exercises 13–28 continuous?

13. $y = \dfrac{1}{x-2} - 3x$ **14.** $y = \dfrac{1}{(x+2)^2} + 4$

Solution

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$ 14. Discontinuous only when $(x+2)^2 = 0 \Rightarrow x = -2$

17. $y = |x - 1| + \sin x$ **18.** $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

Solution

17. Continuous everywhere. ($|x - 1| + \sin x$ defined for all x; limits exist and are equal to function values.)

18. Continuous everywhere. ($|x| + 1 \neq 0$ for all x; limits exist and are equal to function values.)

Find the limits in Exercises, and Are the functions continuous at the point being approached.

29. $\lim_{x \to \pi} \sin(x - \sin x)$

30. $\lim_{t \to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

Solution

29. $\lim_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$.

30. $\lim_{t \to 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right) = \sin\left(\frac{\pi}{2} \cos(\tan(0))\right) = \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.

35. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

Solution

35. $g(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = x + 3,\ x \neq 3 \ \Rightarrow\ g(3) = \lim\limits_{x \to 3} (x+3) = 6$

36. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

Solution

36. $h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t + 5,\ t \neq 2 \ \Rightarrow\ h(2) = \lim\limits_{t \to 2} (t+5) = 7$

39. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

Solution

39. As defined, $\lim\limits_{x \to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim\limits_{x \to 3^+} (2a)(3) = 6a$. For f(x) to be continuous we must have $6a = 8 \ \Rightarrow\ a = \frac{4}{3}$.

CHAPTER Five

3.1 Derivative

Definition: The function f ̄ defined by the formula

$$f^{-}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Is called the derivative with respect to x of the function f. The domain of f consists of all x for which the limit exists.

DEFINITION Derivative Function
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Calculating Derivatives from the Definition

$$\frac{d}{dx} f(x)$$

EXAMPLE 1 Applying the Definition

Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution Here we have $f(x) = \dfrac{x}{x-1}$

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\
&= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.
\end{aligned}$$

Solved questions

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$

Solution

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
$= -2x - h$

Step 3: $f'(x) = \lim_{h\to 0}(-2x-h) = -2x$; $f'(-3) = 6, f'(0) = 0, f'(1) = -2$

2. $F(x) = (x-1)^2 + 1; \quad F'(-1), F'(0), F'(2)$

Solution

2. $F(x) = (x-1)^2 + 1$ and $F(x+h) = (x+h-1)^2 + 1 \Rightarrow F'(x) = \lim_{h\to 0} \frac{[(x+h-1)^2+1]-[(x-1)^2+1]}{h}$
$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h} = \lim_{h\to 0} \frac{2xh+h^2-2h}{h} = \lim_{h\to 0}(2x+h-2)$
$= 2(x-1)$; $F'(-1) = -4, F'(0) = -2, F'(2) = 2$

3. $g(t) = \frac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$

Solution

3. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2\cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2\cdot t^2\cdot h} = \frac{-2th-h^2}{(t+h)^2\,t^2h}$
$= \frac{h(-2t-h)}{(t+h)^2\,t^2h} = \frac{-2t-h}{(t+h)^2\,t^2}$

Step 3: $g'(t) = \lim_{h\to 0} \frac{-2t-h}{(t+h)^2\,t^2} = \frac{-2t}{t^2\cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2, g'(2) = -\frac{1}{4}, g'\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$

4. $k(z) = \frac{1-z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$

5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$

6. $r(s) = \sqrt{2s+1}; \quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives

7. $\frac{dy}{dx}$ if $y = 2x^3$ **8.** $\frac{dr}{ds}$ if $r = \frac{s^3}{2} + 1$

Solution

7. $y = f(x) = 2x^3$ and $f(x+h) = 2(x+h)^3 \Rightarrow \frac{dy}{dx} = \lim_{h \to 0} \frac{2(x+h)^3 - 2x^3}{h} = \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h}$

$$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} = \lim_{h \to 0} \frac{h(6x^2 + 6xh + 2h^2)}{h} = \lim_{h \to 0} (6x^2 + 6xh + 2h^2) = 6x^2$$

8. $r = \frac{s^3}{2} + 1 \Rightarrow \frac{dr}{ds} = \lim_{h \to 0} \frac{\left[\frac{(s+h)^3}{2} + 1\right] - \left[\frac{s^3}{2} + 1\right]}{h} = \frac{1}{2} \lim_{h \to 0} \frac{[(s+h)^3 + 2] - [s^3 + 2]}{h}$

$$= \frac{1}{2} \lim_{h \to 0} \frac{s^3 + 3s^2h + 3sh^2 + h^3 + 2 - s^3 - 2}{h} = \frac{1}{2} \lim_{h \to 0} \frac{h[3s^2 + 3sh + h^2]}{h} = \frac{1}{2} \lim_{h \to 0} (3s^2 + 3sh + h^2) = \frac{3}{2} s^2$$

9. $\frac{ds}{dt}$ if $s = \frac{t}{2t+1}$

10. $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$

11. $\frac{dp}{dq}$ if $p = \frac{1}{\sqrt{q+1}}$

12. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{3w-2}}$

3.2 Differentiation Rules

Laws of derivatives:

the derivative of a constant is zero.

1. $(x^n)' = nx^{n-1}$
2. $(cf(x))' = cf(x)'$
3. $(f(x) \mp g(x))' = f(x)' + g(x)'$
4. $(f(x).g(x))' = f(x).g(x)' + f(x)' g(x)$
5. $[(f(x))^n]' = n(f(x))^{n-1} f(x)'$
6. $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\, f(x)' - f(x) g(x)'}{g(x)^2}$

EXAMPLE 1

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.8). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

EXAMPLE 3

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3\cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.9).

(b) **A useful special case**

The derivative of the negative of a differentiable function u is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\cdot\frac{d}{dx}(u) = -\frac{du}{dx}.$$

Proof of Rule 3

$$\begin{aligned}
\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition with } f(x) = cu(x)\\
&= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} && \text{Limit property}\\
&= c\frac{du}{dx} && u \text{ is differentiable.}
\end{aligned}$$

EXAMPLE 4 Derivative of a Sum

$$\begin{aligned}
y &= x^4 + 12x\\
\frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)\\
&= 4x^3 + 12
\end{aligned}$$

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}
\frac{d}{dx}[u(x) + v(x)] &= \lim_{h\to 0}\frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}\\
&= \lim_{h\to 0}\left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]\\
&= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} + \lim_{h\to 0}\frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.
\end{aligned}$$

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\frac{d}{dx}\left[\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right] = \frac{1}{x}\left(2x - \frac{1}{x^2}\right) + \left(x^2 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ by Example 3, Section 2.7.

$$= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3}$$

$$= 1 - \frac{2}{x^3}.$$

■

Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right]$$

$$= \lim_{h \to 0} u(x+h) \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}.$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

■

EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}.$$

Proof of Rule 6

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}.$$

EXAMPLE 11

(a) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$

(b) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$

Proof of Rule 7 The proof uses the Quotient Rule. If n is a negative integer, then $n = -m$, where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{Quotient Rule with } u = 1 \text{ and } v = x^m$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} \qquad \text{Since } m > 0, \frac{d}{dx}(x^m) = mx^{m-1}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}. \qquad \text{Since } -m = n \qquad \blacksquare$$

Second and Higher Order Derivative

$$y' = \frac{dy}{dx} \qquad \text{(first order)}$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \qquad \text{(second order)}$$

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \qquad \text{(third order)}$$

$$y^n = \frac{d}{dx}(y)^{n-1} \qquad \text{(nth order)}$$

EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

Application

Velocity is the derivative of the distance. Speed is the absolute value of velocity.

Speed = |velocity|
Speed = $|v(t)| = |ds/dt|$

Acceleration is the derivative of velocity with respect to time.

$$a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

Example

The distance of a ball falls freely from rest is proportional with time as $s=4.9t^2$

a- How long did it take the ball bearing to fall the first 14.7m?

b- What is the velocity, speed and acceleration after 2 second?

Solution

a- $s = 4.9t^2$

$14.7=4.9t^2$ so $t = \pm\sqrt{3}$ second

$t= \sqrt{3}$ (time increase from t=0 so we ignore the negative root)

b- velocity at any time

$$v(t)=\frac{ds}{dt}=9.8t$$

Velocity after 2 second = v(2) = 19.6 m/s

Speed = |19.6| = 19.6 m/s

Acceleration at any time

$$a(t)=\frac{d^2s}{dt^2}=9.8m/s^2$$

Acceleration after 2 second = 9.8 m/s^2

Example

A dynamite blast blows heavy rock straight up with a lunch velocity of 160 ft/sec. It reaches a height of s=160t -16t^2 after time (sec)

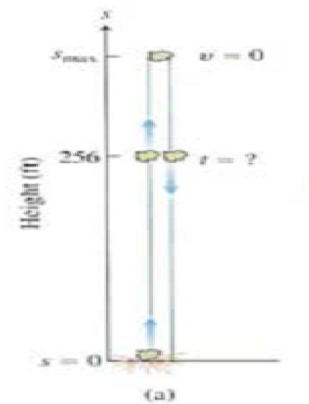
a- How height does the rock ago?

b- what are thw velocity and speed of the rock when its 256 ft above the ground on the way up ? on the way down?

c- what is the acceleration of the rock at any time t during its flight (after the blast)?

d- when does the rock hit the ground again?

Solution



a- $v = \frac{ds}{dt} = 160 - 32t$ ft/sec

at v=0 t=5 sec

$S_{max} = s(5) = 400$ft

b- $s(t) = 160 - 16\ t^2$

at s = 256 ft then the time will be 2 and 8 second

v(2) = 160 – 32 (2) = 96 ft/s

v(8) = 160 - 32 (8) = - 96 ft /s

At both instants, the rocks speed is 96 ft/s

c- $a = \frac{dv}{dt} = -32\ ft/\sec^2$ (the acceleration is always downword)

d- at s=0 then time will be 10 sec

Solved questions

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$ **2.** $y = x^2 + x + 8$

Solution

$$1.\quad y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$$

$$2.\quad y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$$

3. $s = 5t^3 - 3t^5$ **4.** $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \frac{4x^3}{3} - x$ **6.** $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

7. $w = 3z^{-2} - \frac{1}{z}$ **8.** $s = -2t^{-1} + \frac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$ **10.** $y = 4 - 2x - x^{-3}$

11. $r = \frac{1}{3s^2} - \frac{5}{2s}$ **12.** $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate

13. $y = (3 - x^2)(x^3 - x + 1)$ **14.** $y = (x - 1)(x^2 + x + 1)$

Solution

13. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$

(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

14. (a) $y = (x - 1)(x^2 + x + 1) \Rightarrow y' = (x - 1)(2x + 1) + (x^2 + x + 1)(1) = 3x^2$

(b) $y = (x - 1)(x^2 + x + 1) = x^3 - 1 \Rightarrow y' = 3x^2$

15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$ **16.** $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \frac{2x + 5}{3x - 2}$ **18.** $z = \frac{2x + 1}{x^2 - 1}$

Solution

17. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
$= \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2} = \frac{6x - 4 - 6x - 15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

18. $z = \frac{2x+1}{x^2-1} \Rightarrow \frac{dz}{dx} = \frac{(x^2-1)(2) - (2x+1)(2x)}{(x^2-1)^2} = \frac{2x^2 - 2 - 4x^2 - 2x}{(x^2-1)^2} = \frac{-2x^2 - 2x - 2}{(x^2-1)^2} = \frac{-2(x^2 + x + 1)}{(x^2-1)^2}$

19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$ **20.** $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$ **22.** $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$ **24.** $u = \dfrac{5x + 1}{2\sqrt{x}}$

25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$ **26.** $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$ **28.** $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \dfrac{x^3 + 7}{x}$ **32.** $s = \dfrac{t^2 + 5t - 1}{t^2}$

Solution

31. $y = \frac{x^3+7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$

32. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

33. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$ **34.** $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

35. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$ **36.** $w = (z + 1)(z - 1)(z^2 + 1)$

3.5 The Chain Rule and Parametric Equations

The Chain Rule is one of the most important and widely used rules of differentiation. This section describes the rule and how to use it.

If $y = f(t)$ and $x = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

And

$$\frac{d^2y}{dx^2} = \frac{dy^{-}}{dx} / \frac{dx}{dt} \qquad \text{or} \qquad \frac{d^2y}{dx^2} = \frac{dy^{-}}{dx} \cdot \frac{dt}{dx}$$

EXAMPLE 1 Relating Derivatives

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. How are the derivatives of these functions related?

Solution We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example

Find d^2y / dx^2 if $x = t - t^2$ and $y = t - t^3$

Solution

$$y^{-} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\frac{dy^{-}}{dt} = \frac{2-6t+6t^2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy^{-}/dt}{dx/dt} = \frac{2-6t+6t^2}{(1-2t)^3}$$

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

EXAMPLE 4 Differentiating from the Outside In

Differentiate $\sin(x^2 + x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\text{inside}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside}\\\text{left alone}}}\cdot\underbrace{(2x+1)}_{\substack{\text{derivative of}\\\text{the inside}}}$$

The Chain Rule with Powers of a Function

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}\left(u^n\right) = nu^{n-1}$$

EXAMPLE 6 Applying the Power Chain Rule

(a)
$$\begin{aligned}\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6\frac{d}{dx}\left(5x^3 - x^4\right) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7\\ &= 7(5x^3 - x^4)^6(5\cdot 3x^2 - 4x^3)\\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3)\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1}\\ &= -1(3x-2)^{-2}\frac{d}{dx}(3x-2) && \text{Power Chain Rule with } u = 3x - 2, n = -1\\ &= -1(3x-2)^{-2}(3)\\ &= -\frac{3}{(3x-2)^2}\end{aligned}$$

Solved questions

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$ **2.** $y = 2u^3, \quad u = 8x - 1$

Solution

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x-1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= 6(8x-1)^2 \cdot 8 = 48(8x-1)^2$

3. $y = \sin u, \quad u = 3x + 1$ **4.** $y = \cos u, \quad u = -x/3$

Solutions

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x+1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= (\cos(3x+1))(3) = 3\cos(3x+1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin\left(\frac{-x}{3}\right)$; $g(x) = \frac{-x}{3} \Rightarrow g'(x) = -\frac{1}{3}$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= -\sin\left(\frac{-x}{3}\right) \cdot \left(\frac{-1}{3}\right) = \frac{1}{3}\sin\left(\frac{-x}{3}\right)$

5. $y = \cos u, \quad u = \sin x$ **6.** $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$ **8.** $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$ **10.** $y = (4 - 3x)^9$

Solution

9. With $u = (2x+1)$, $y = u^5$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \cdot 2 = 10(2x+1)^4$

10. With $u = (4-3x)$, $y = u^9$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 \cdot (-3) = -27(4-3x)^8$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$ **12.** $y = \left(\frac{x}{2} - 1\right)^{-10}$

Solution

11. With $u = \left(1 - \frac{x}{7}\right)$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$

12. With $u = \left(\frac{x}{2} - 1\right)$, $y = u^{-10}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -10u^{-11} \cdot \left(\frac{1}{2}\right) = -5\left(\frac{x}{2} - 1\right)^{-11}$

13. $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$ **14.** $y = \left(\dfrac{x}{5} + \dfrac{1}{5x}\right)^5$

15. $y = \sec(\tan x)$ **16.** $y = \cot\left(\pi - \dfrac{1}{x}\right)$

17. $y = \sin^3 x$ **18.** $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3 - t}$ **20.** $q = \sqrt{2r - r^2}$

21. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$

Solution

$$19.\ \ p = \sqrt{3-t} = (3-t)^{1/2} \Rightarrow \frac{dp}{dt} = \frac{1}{2}(3-t)^{-1/2} \cdot \frac{d}{dt}(3-t) = -\frac{1}{2}(3-t)^{-1/2} = \frac{-1}{2\sqrt{3-t}}$$

22. $s = \sin\left(\dfrac{3\pi t}{2}\right) + \cos\left(\dfrac{3\pi t}{2}\right)$

23. $r = (\csc\theta + \cot\theta)^{-1}$ **24.** $r = -(\sec\theta + \tan\theta)^{-1}$

25. $y = x^2\sin^4 x + x\cos^{-2} x$ **26.** $y = \dfrac{1}{x}\sin^{-5} x - \dfrac{x}{3}\cos^3 x$

27. $y = \dfrac{1}{21}(3x - 2)^7 + \left(4 - \dfrac{1}{2x^2}\right)^{-1}$

28. $y = (5 - 2x)^{-3} + \dfrac{1}{8}\left(\dfrac{2}{x} + 1\right)^4$

29. $y = (4x + 3)^4(x + 1)^{-3}$ **30.** $y = (2x - 5)^{-1}(x^2 - 5x)^6$

31. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$ **32.** $k(x) = x^2\sec\left(\dfrac{1}{x}\right)$

33. $f(\theta) = \left(\dfrac{\sin\theta}{1 + \cos\theta}\right)^2$ **34.** $g(t) = \left(\dfrac{1 + \cos t}{\sin t}\right)^{-1}$

35. $r = \sin(\theta^2)\cos(2\theta)$ **36.** $r = \sec\sqrt{\theta}\tan\left(\dfrac{1}{\theta}\right)$

37. $q = \sin\left(\dfrac{t}{\sqrt{t+1}}\right)$ **38.** $q = \cot\left(\dfrac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$ **40.** $y = \sec^2 \pi t$

41. $y = (1 + \cos 2t)^{-4}$ **42.** $y = (1 + \cot(t/2))^{-2}$

Solution

$$39.\ \ y = \sin^2(\pi t - 2) \Rightarrow \frac{dy}{dt} = 2\sin(\pi t - 2)\cdot\frac{d}{dt}\sin(\pi t - 2) = 2\sin(\pi t - 2)\cdot\cos(\pi t - 2)\cdot\frac{d}{dt}(\pi t - 2)$$
$$= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)$$

43. $y = \sin(\cos(2t - 5))$ **44.** $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$ **46.** $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

47. $y = \sqrt{1 + \cos(t^2)}$ **48.** $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$ **50.** $y = \left(1 - \sqrt{x}\right)^{-1}$

51. $y = \frac{1}{9}\cot(3x - 1)$ **52.** $y = 9\tan\left(\frac{x}{3}\right)$

Solution

$$49.\ \ y = \left(1 + \tfrac{1}{x}\right)^3 \Rightarrow y' = 3\left(1 + \tfrac{1}{x}\right)^2\left(-\tfrac{1}{x^2}\right) = -\tfrac{3}{x^2}\left(1 + \tfrac{1}{x}\right)^2 \Rightarrow y'' = \left(-\tfrac{3}{x^2}\right)\cdot\tfrac{d}{dx}\left(1 + \tfrac{1}{x}\right)^2 - \left(1 + \tfrac{1}{x}\right)^2\cdot\tfrac{d}{dx}\left(\tfrac{3}{x^2}\right)$$
$$= \left(-\tfrac{3}{x^2}\right)\left(2\left(1 + \tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right)\right) + \left(\tfrac{6}{x^3}\right)\left(1 + \tfrac{1}{x}\right)^2 = \tfrac{6}{x^4}\left(1 + \tfrac{1}{x}\right) + \tfrac{6}{x^3}\left(1 + \tfrac{1}{x}\right)^2 = \tfrac{6}{x^3}\left(1 + \tfrac{1}{x}\right)\left(\tfrac{1}{x} + 1 + \tfrac{1}{x}\right)$$
$$= \tfrac{6}{x^3}\left(1 + \tfrac{1}{x}\right)\left(1 + \tfrac{2}{x}\right)$$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x), \quad x = 2$ **b.** $f(x) + g(x), \quad x = 3$

c. $f(x) \cdot g(x), \quad x = 3$ **d.** $f(x)/g(x), \quad x = 2$

e. $f(g(x)), \quad x = 2$ **f.** $\sqrt{f(x)}, \quad x = 2$

g. $1/g^2(x), \quad x = 3$ **h.** $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

Solution

$$59.\ (a)\quad y = 2f(x) \Rightarrow \frac{dy}{dx} = 2f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = 2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

$$(b)\quad y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x) \Rightarrow \left.\frac{dy}{dx}\right| = f'(3) + g'(3) = 2\pi + 5$$

Implicite Function Differentiation

We can find the derivative of implicit functions in two steps:

Step 1: Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x

Step 2: Solve for dy/dx

Example

Find dy/dx for $y^2 + 5xy - 6x^2 = 0$

Solution

$$2y\frac{dy}{dx} + 5x\frac{dy}{dx} + 5y - 12x = 0$$

$$\frac{dy}{dx} = \frac{12x - 5y}{2y + 5x}$$

Example

Assume that the radius r and the height of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt and dh/dt.

Solution

$$V = \frac{\pi}{3} r^2 h$$
$$\frac{dV}{dt} = \frac{\pi}{3}(r^2 \frac{dh}{dt} + 2r\frac{dr}{dt} h$$
$$\frac{dV}{dt} = \frac{\pi}{3}(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt})$$

EXAMPLE 4 Tangent and Normal to the Folium of Descartes

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.41).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$ Differentiate both sides with respect to x.
$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$ Treat xy as a product and y as a function of x.
$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$
$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$
$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$ Solve for dy/dx.

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope 4/5:

$$y = 4 + \frac{4}{5}(x - 2)$$
$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$
$$y = -\frac{5}{4}x + \frac{13}{2}.$$

The quadratic formula enables us to solve a second-degree equation like $y^2 - 2xy + 3x^2 = 0$ for y in terms of x. There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If this formula is used to solve the equation $x^3 + y^3 = 9xy$ for y in terms of x, then three functions determined by the equation are

$$y = f(x) = \sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} + \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}$$

and

$$y = \frac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} - \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}\right)\right].$$

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives. Here is an example.

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

Solved question

Find dy/dx in Exercises 1-10.

1. $y = x^{9/4}$ **2.** $y = x^{-3/5}$

Solution

1. $y = x^{9/4} \Rightarrow \frac{dy}{dx} = \frac{9}{4}x^{5/4}$

2. $y = x^{-3/5} \Rightarrow \frac{dy}{dx} = -\frac{3}{5}x^{-8/5}$

3. $y = \sqrt[3]{2x}$ **4.** $y = \sqrt[4]{5x}$

5. $y = 7\sqrt{x + 6}$ **6.** $y = -2\sqrt{x - 1}$

7. $y = (2x + 5)^{-1/2}$ **8.** $y = (1 - 6x)^{2/3}$

9. $y = x(x^2 + 1)^{1/2}$ **10.** $y = x(x^2 + 1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11-18.

11. $s = \sqrt[7]{t^2}$ **12.** $r = \sqrt[4]{\theta^{-3}}$

Solution

11. $s = \sqrt[7]{t^2} = t^{2/7} \Rightarrow \frac{ds}{dt} = \frac{2}{7}t^{-5/7}$

12. $r = \sqrt[4]{\theta^{-3}} = \theta^{-3/4} \Rightarrow \frac{dr}{d\theta} = -\frac{3}{4}\theta^{-7/4}$

13. $y = \sin[(2t+5)^{-2/3}]$ **14.** $z = \cos[(1-6t)^{2/3}]$

15. $f(x) = \sqrt{1-\sqrt{x}}$ **16.** $g(x) = 2(2x^{-1/2}+1)^{-1/3}$

17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$ **18.** $k(\theta) = (\sin(\theta+5))^{5/4}$

Use implicit differentiation to find dy/dx in Exercises 19-32.

19. $x^2y + xy^2 = 6$ **20.** $x^3 + y^3 = 18xy$

Solution

19. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$

Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}(x^2+2xy) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$

20. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 18y + 18x\frac{dy}{dx} \Rightarrow (3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y-x^2}{y^2-6x}$

21. $2xy + y^2 = x + y$ **22.** $x^3 - xy + y^3 = 1$

23. $x^2(x-y)^2 = x^2 - y^2$ **24.** $(3xy+7)^2 = 6y$

25. $y^2 = \frac{x-1}{x+1}$ **26.** $x^2 = \frac{x-y}{x+y}$

27. $x = \tan y$ **28.** $xy = \cot(xy)$

29. $x + \tan(xy) = 0$ **30.** $x + \sin y = xy$

31. $y\sin\left(\frac{1}{y}\right) = 1 - xy$ **32.** $y^2\cos\left(\frac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33-36.

33. $\theta^{1/2} + r^{1/2} = 1$ **34.** $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

solution

33. $\theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta}\left[\frac{1}{2\sqrt{r}}\right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

34. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$

In Exercises 37-42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$ **38.** $x^{2/3} + y^{2/3} = 1$

Solution

37. $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow \frac{dy}{dx} = y' = -\frac{x}{y}$; now to find $\frac{d^2y}{dx^2}$, $\frac{d}{dx}(y') = \frac{d}{dx}\left(-\frac{x}{y}\right)$

$\Rightarrow y'' = \frac{y(-1)+xy'}{y^2} = \frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$ since $y' = -\frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{-y^2-x^2}{y^3} = \frac{-y^2-(1-y^2)}{y^3} = \frac{-1}{y^3}$

38. $x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right] = -\frac{2}{3}x^{-1/3} \Rightarrow y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y'' = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y' + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}x^{-2/3}y^{-1/3} + \frac{1}{3}y^{1/3}x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$

39. $y^2 = x^2 + 2x$ **40.** $y^2 - 2x = 1 - 2y$

41. $2\sqrt{y} = x - y$ **42.** $xy + y^2 = 1$

43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

In Exercises 45 and 46, find the slope of the curve at the given points.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

46. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

Solution

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1) \Rightarrow 2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2 \Rightarrow 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -2 - 2x$

$\Rightarrow \frac{dy}{dx}(2y - 4y^3) = -2 - 2x \Rightarrow \frac{dy}{dx} = \frac{x+1}{2y^3-y} \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

46. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1) \Rightarrow 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2(x - y)\left(1 - \frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}\left[2y(x^2 + y^2) + (x - y)\right] = -2x(x^2 + y^2) + (x - y) \Rightarrow \frac{dy}{dx} = \frac{-2x(x^2+y^2)+(x-y)}{2y(x^2+y^2)+(x-y)} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,0)} = -1$

and $\left.\frac{dy}{dx}\right|_{(1,-1)} = 1$

4.3 Increasing Functions and Decreasing Functions.

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval.

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

EXAMPLE 1 Using the First Derivative Test for Monotonic Functions

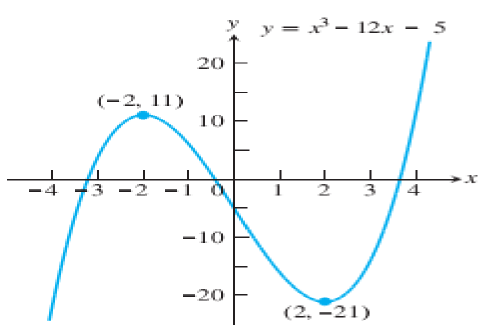
Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$. The
$(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$

Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



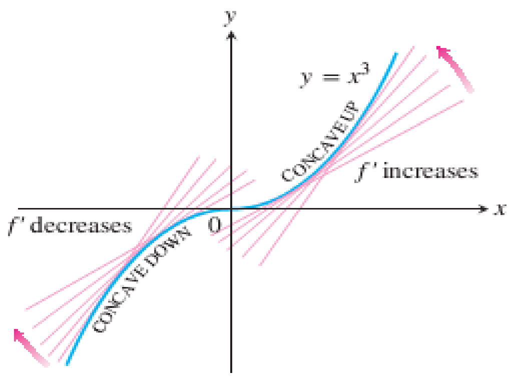
4.4 Concavity and Curve Sketching

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.



The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE 1 Applying the Concavity Test

(a) The curve $y = x^3$ (Figure 4.25) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.26) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determining Concavity

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.27). ■

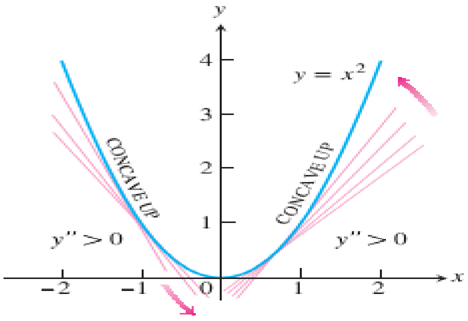


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

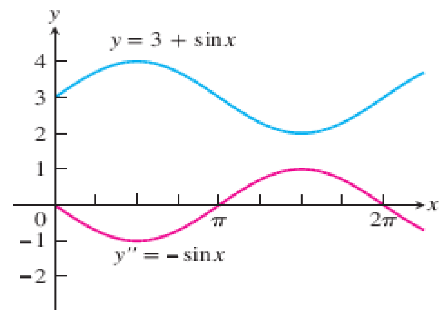


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

EXAMPLE 5 Studying Motion Along a Line

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	+	−	+
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest), at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	−	+
Graph of s	concave down	concave up

Second Derivative Test for Local Extreme

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

EXAMPLE 6 Using f' and f'' to Graph f

Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.
(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
(c) Find where the graph of f is concave up and where it is concave down.
(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

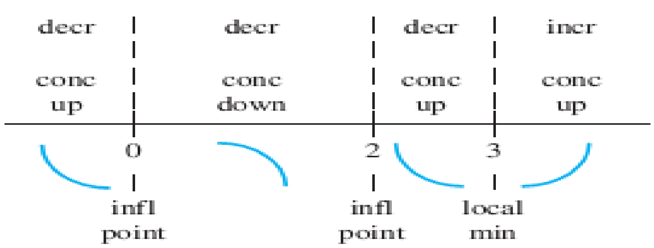
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

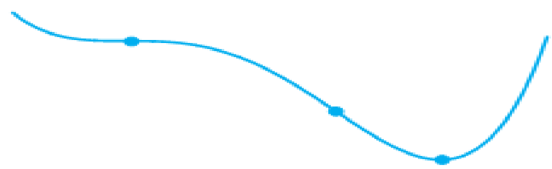
(d) Summarizing the information in the two tables above, we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is



General shape.



(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f. ■

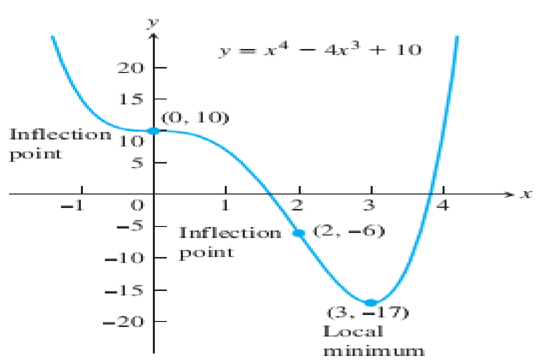


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

EXAMPLE 7 Using the Graphing Strategy

Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.4).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

Expanding numerator

$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}.$$

Dividing by x^2

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).
7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

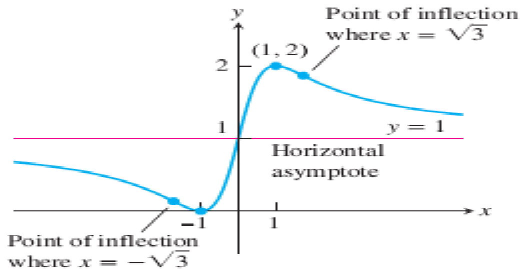


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 7).

<u>4.6</u> Application of Derivatives on Limits:

<u>L'Hôpital's Rule</u>

$$if \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}\ or\ \frac{\infty}{\infty} \qquad ,\text{then}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example: find

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}. \quad \blacksquare$$

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\Bigg|_{x=0} = 2$

(b) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \frac{\dfrac{1}{2\sqrt{1 + x}}}{1}\Bigg|_{x=0} = \frac{1}{2}$

(a) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}$ $\qquad \dfrac{0}{0}$

$$= \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1 + x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Example

Find $\lim_{x\to 0}\frac{3x-\sin x}{x}$

Solution

$$=\lim_{x\to 0}\frac{3-\cos x}{1}=2$$

Example

Find $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Solution

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$$
$$=\lim_{x\to 0}\frac{\sin x}{6x}$$
$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

Example

$$\lim_{x\to \pi}\frac{\tan x}{1+\tan x}$$
$$=\lim_{x\to \pi}\frac{\sec^2 x}{\sec^2 x}=1$$

Example

$$\lim_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})$$
$$=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$$
$$=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}=\frac{0}{2}=0$$

Example

$$\lim_{x\to 0}\left(\frac{e^{2x+\ln x}}{x+x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{xe^{2x}}{x+x^2}\right)$$

$$=\lim_{x\to 0}\frac{2e^{2x}.x+e^{2x}}{1+2x}=\frac{1}{1}=1$$

Theorem

If $\lim\limits_{x\to a}\ln\ f(x)=L$

Then $\lim\limits_{x\to a}\ f(x)=\lim\limits_{x\to a}e^{\ln\ f(x)}=e^{L}$

Example

$$\lim_{x\to o}(1-x^2)^{-\frac{1}{x^2}}$$

$$=\ f(x)=(1-x^2)^{-\frac{1}{x^2}}$$

$$\ln\ f(x)=-\frac{1}{x^2}\ln(1-x^2)$$

$$\therefore \lim_{x\to 0}\ln\ f(x)=-\lim_{x\to 0}\frac{\ln(1-x^2)}{x^2}$$

$$=-\lim_{x\to 0}\frac{\frac{-2x}{1-x^2}}{2x}$$

$$\therefore \lim_{x\to 0}\ln\ f(x)=+\lim_{x\to 0}\frac{1}{1-x^2}=1$$

$$\therefore \lim_{x\to 0}\ f(x)=e^{1}$$

H.W

Find

1- $\lim\limits_{x\to\pi/3}\dfrac{\cos x-0.5}{x-\pi/3}$ 2- $\lim\limits_{x\to\infty}x-\sqrt{x^2+x}$

Derivative of Exponential and Logarithmic Function

If u = f(x)

1- $y = b^u$ $\qquad \dfrac{dy}{dx} = b^u.\ln b.\dfrac{du}{dx}$

2- $y = e^u$ $\qquad \dfrac{dy}{dx} = e^u.\dfrac{du}{dx}$

3- $y = \log_b u$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{u}.\dfrac{1}{\ln b}.\dfrac{du}{dx}$

4- $y = \ln u$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{u}.\dfrac{du}{dx}$

Example

$$y = x^{\sin x} \Rightarrow \ln y = \sin x.\ln x$$

$$\frac{1}{y}.\frac{dy}{dx} = \cos x.\ln x + \sin x.\frac{1}{x}$$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x.\ln x\right)$$

Example

$$y = \ln\sqrt{\frac{1+x}{1-x}} \Rightarrow \ln y = \frac{1}{2}(\ln(1+x) - \ln(1-x))$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

Example

$$\tan y = e^x + \ln x$$

$$\sec^2 y.\frac{dy}{dx} = e^x + \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\frac{xe^x + 1}{x}\right)$$

Example

$$y = 3^{-x}$$

$$\frac{dy}{dx} = -3^{-x}.\ln x$$

Example

$$y = \pi^{\sin x}$$

$$\frac{dy}{dx} = \pi^{\sin x}.\ln \pi.\cos x$$

H.W

Find the derivative for

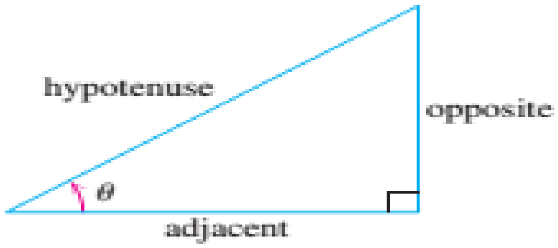
a) $y = x^{x^2}$

b) $y = \ln(x^2 + 3)$

c) $y = e^{\sin x}$

CHAPTER Six

1.6 Trigonometric Functions



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

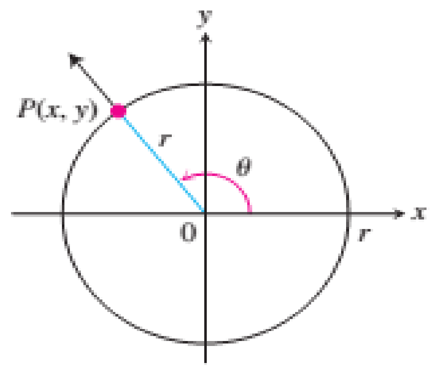


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

The define the trigonometric functions in terms of the coordinates of the point P(x, y) where the angle's terminal ray intersects the circle (Figure 1.68).

sine: $\sin\theta = \dfrac{y}{r}$ cosecant: $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ secant: $\sec\theta = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{y}{x}$ cotangent: $\cot\theta = \dfrac{x}{y}$

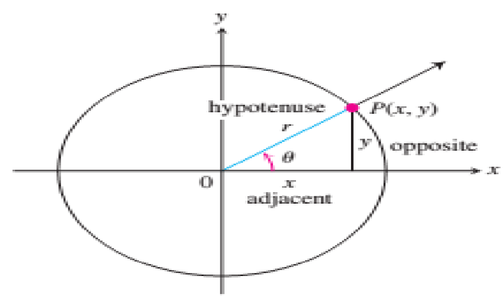


FIGURE 1.69 The new and old definitions agree for acute angles.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

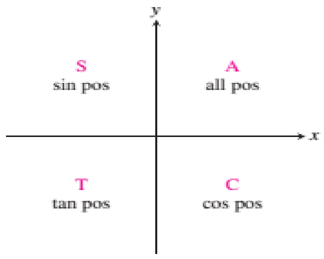
$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}}$$

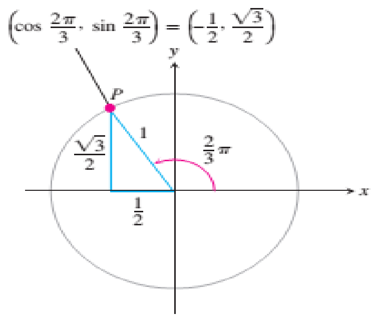
Conversion formula:

1 degree = π / 180 ($\approx$ 0.02) radian

1 radian = 180 / π ($\approx$ 57) degree

The CAST rule (Figure 1.70) is useful for remembering when the basic trigonometric functions are positive or negative





The CAST rule, remembered by the statement "**A**ll **S**tudents **T**ake **C**alculus," tells which trigonometric functions are positive in each quadrant

Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

Periods of Trigonometric Functions

Period π :

$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

Period 2π :

$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$

Periodicity

A function f is periodic if there is a positive number p such that f (x+p) = f (x). the smallest such value of p is the period of f.

$\cos(\theta + 2\pi) = \cos\theta$, $\sin(\theta + 2\pi) = \sin\theta$

$\tan(\theta + 2\pi) = \tan\theta$, $\sec(\theta + 2\pi) = \sec\theta$

$\sec(\theta + 2\pi) = \csc\theta$, $\cot(\theta + 2\pi) = \cot\theta$

similarly
$\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$ and so on....

Graphs of trigonometric functions

When we graph trigonometric functions in the coordinate plane, we denote the independent variable (radians) by x instead of θ.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

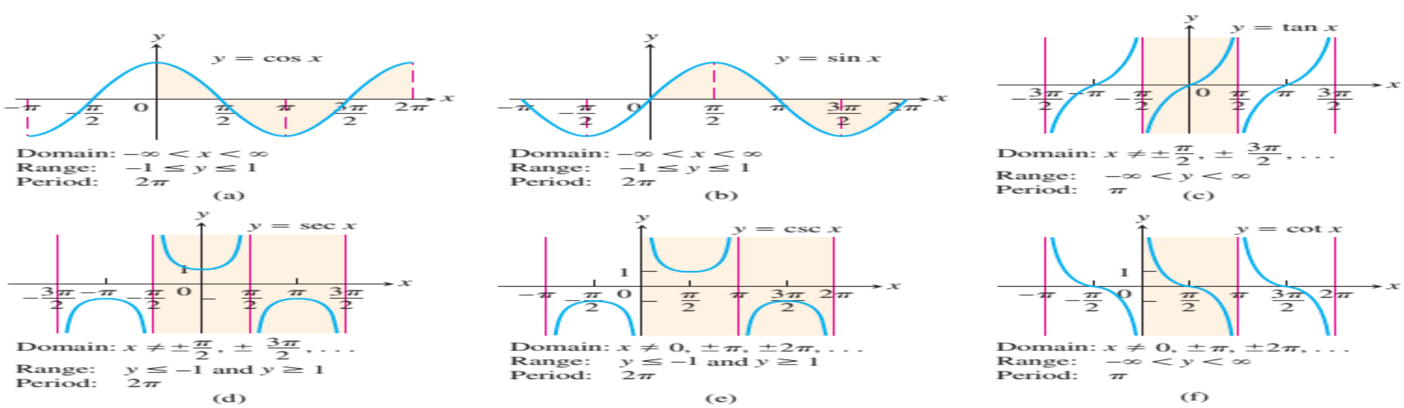


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Even and odd trigonometric functions

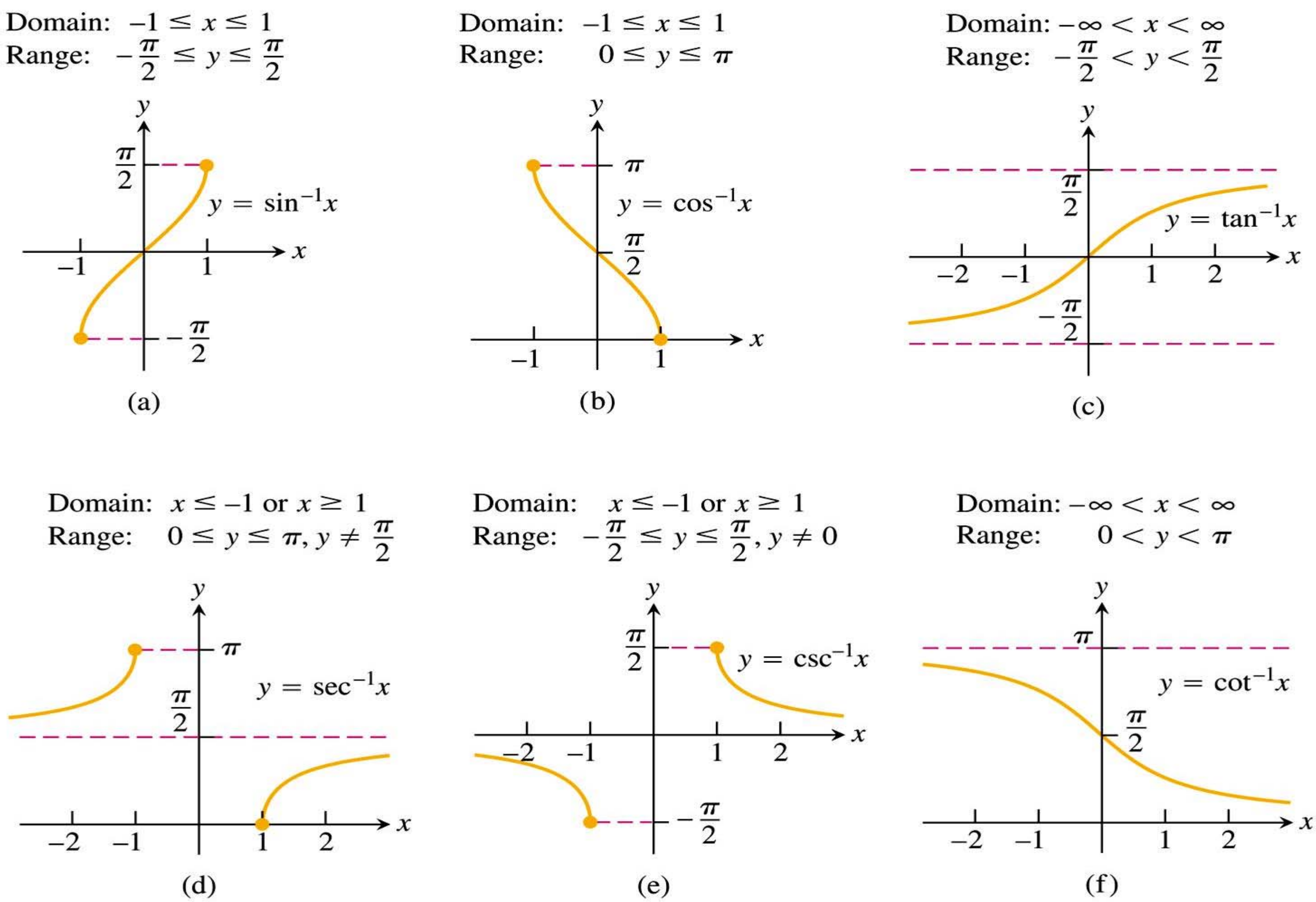
The graph in the above figures suggest that cos θ and sin θ are even functions because their graphs are symmetric about the y-axis. The other four basic trigonometric functions odd.

cos (-θ) = cos θ
sin (-θ) = -sin θ
sec(-θ)=1/cos (-θ)= 1/cos θ = sec θ

Inverse trigonometric functions and their graphs

Inverse Trigonometric functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions, written $\cos^{-1} z$, $\cot^{-1} z$, $\csc^{-1} z$, $\sec^{-1} z$, $\sin^{-1} z$, and $\tan^{-1} z$.



Sec^{-1}=1/Cos = Cos,

Csc^{-1}=1/Sin = Sin,

Cot^{-1}=1/Tan = Tan

<u>IMPORTANT:</u> Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

$\sin(-x) = -\sin x$ $\qquad$ $\cos(-x) = \cos x$ $\qquad$ $\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$ $\qquad$ $\sec(-x) = \sec x$ $\qquad$ $\cot(-x) = -\cot x$

$\sin(x \pm \pi) = -\sin x$ $\qquad$ $\cos(x \pm \pi) = -\cos x$ $\qquad$ $\tan(x \pm \pi) = \tan x$

$\sec(x \pm \pi) = -\sec x$ $\qquad$ $\csc(x \pm \pi) = -\csc x$ $\qquad$ $\cot(x \pm \pi) = \cot x$

EXAMPLES:

(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(f) $\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

- **Basic idea:** To find $\sec^{-1} 2$, we ask "what angle has secant equal to 2?" The answer is 60°. As a result we say that $\sec^{-1} 2 = 60°$. In radians this is $\sec^{-1} 2 = \pi/3$.
- More: There are actually many angles that have secant equal to 2. We are really asking "what is the simplest, most basic angle that has secant equal to 2?" As before, the answer is 60°. Thus $\sec^{-1} 2 = 60°$ or $\sec^{-1} 2 = \pi/3$.
- Details: What is $\sec^{-1}(-2)$? Do we choose 120°, –120°, 240° , or some other angle? The answer is 120°. With inverse secant, we select the angle on the top half of the unit circle. Thus $\sec^{-1}(-2) = 120°$ or $\sec^{-1}(-2) = 2\pi/3$.

Note: sec 90° is undefined, so 90° is not in the range of sec-1

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

Solution: We have

$$\sec^{-1} 1 = 0, \qquad \sec^{-1}(-1) = \pi, \qquad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \qquad \sec \pi = -1, \qquad \sec \frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Note that $\sec \frac{2\pi}{3}$ is also -2, but

$$\sec^{-1}(-2) \neq \frac{2\pi}{3}$$

since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Find

$$\tan^{-1} 0 \qquad \cot^{-1} 0 \qquad \cot^{-1} 1 \qquad \sec^{-1} \sqrt{2} \qquad \csc^{-1} 2 \qquad \csc^{-1} \frac{2}{\sqrt{3}}$$

EXAMPLES: We have

$$\tan^{-1} 0 = 0, \quad \cot^{-1} 0 = \frac{\pi}{2}, \quad \cot^{-1} 1 = \frac{\pi}{4}, \quad \sec^{-1} \sqrt{2} = \frac{\pi}{4}, \quad \csc^{-1} 2 = \frac{\pi}{6}, \quad \csc^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{3}$$

Can use this formula

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

or

$$arc\cot x = \frac{\pi}{2} - arc\tan x$$

EXAMPLES: Evaluate $\sin\left(\arcsin\frac{\pi}{7}\right)$, $\arcsin\left(\sin\frac{\pi}{7}\right)$, and $\arcsin\left(\sin\frac{8\pi}{7}\right)$.

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1,1] \qquad \text{and} \qquad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$

Therefore

$$\sin\left(\arcsin\frac{\pi}{7}\right) = \frac{\pi}{7} \qquad \text{and} \qquad \arcsin\left(\sin\frac{\pi}{7}\right) = \frac{\pi}{7}$$

but

$$\arcsin\left(\sin\frac{8\pi}{7}\right) = \arcsin\left(\sin\left(\frac{\pi}{7}+\pi\right)\right) = \arcsin\left(-\sin\frac{\pi}{7}\right) = -\arcsin\left(\sin\frac{\pi}{7}\right) = -\frac{\pi}{7}$$

EXAMPLES: Evaluate $\cot\left(\arcsin\frac{2}{5}\right)$ and $\sec\left(\arcsin\frac{2}{5}\right)$.

Solution 1: We have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\sin^2\theta}}$$

Since $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$, it follows that $\cos(\arcsin x) \ge 0$. Therefore if $\theta = \arcsin\frac{2}{5}$, then

$$\cot\theta = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$$

hence

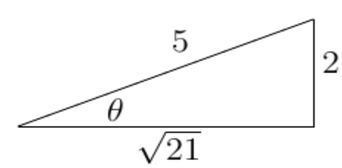
$$\cot\left(\arcsin\frac{2}{5}\right) = \frac{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}}{\sin\left(\arcsin\frac{2}{5}\right)} = \frac{\sqrt{1-\left(\frac{2}{5}\right)^2}}{\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

and

$$\sec\left(\arcsin\frac{2}{5}\right) = \frac{1}{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}} = \frac{1}{\sqrt{1-\left(\frac{2}{5}\right)^2}} = \frac{5}{\sqrt{21}}$$

Solution 2: Put $\theta = \arcsin\frac{2}{5}$, so $\sin\theta = \frac{2}{5}$. Then

$$\cot\left(\arcsin\frac{2}{5}\right) = \cot\theta = \frac{\sqrt{21}}{2} \quad \text{and} \quad \sec\left(\arcsin\frac{2}{5}\right) = \sec\theta = \frac{5}{\sqrt{21}}$$



EXAMPLES: Evaluate, if possible, $\cot\left(\sin^{-1} 2\right)$ and $\sin\left(\tan^{-1} 2\right)$.

We first note that $\sin^{-1} 2$ does not exist, since $2 \notin [-1, 1]$, that is, 2 is not in the domain of $\sin^{-1} x$. Therefore $\cot\left(\sin^{-1} 2\right)$ does not exist.

We will evaluate $\sin\left(\tan^{-1} 2\right)$ in two different ways:

Solution 1: We have

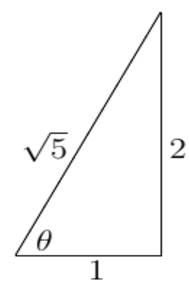
$$\sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

Since $-\pi/2 < \tan^{-1} x < \pi/2$, it follows that $\cos\left(\tan^{-1} x\right) > 0$. Therefore if $\theta = \tan^{-1} 2$, then

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

hence

$$\sin\left(\tan^{-1} 2\right) = \frac{\tan\left(\tan^{-1} 2\right)}{\sqrt{1+\tan^2\left(\tan^{-1} 2\right)}} = \frac{2}{\sqrt{1+2^2}} = \frac{2}{\sqrt{5}}$$



Solution 2: Put $\theta = \tan^{-1} 2 = \tan^{-1}\dfrac{2}{1}$, so $\tan\theta = \dfrac{2}{1}$. Then

$$\sin\left(\tan^{-1} 2\right) = \sin\theta = \frac{2}{\sqrt{5}}$$

EXAMPLES: Evaluate $\sin\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$ and $\cos\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$.

Derivative of Trigonometric Functions

If $u = f(x)$ then

$y = \sin u$	$y^- = \cos u \, .du/dx$
$y = \cos u$	$y^- = -\sin u \, .du/dx$
$y = \tan u$	$y^- = \sec^2 u \, .du/dx$
$y = \cot u$	$y^- = -\csc^2 u \, .du/dx$
$y = \sec u$	$y^- = \sec u \, . \tan u.du/dx$
$y = \csc u$	$y^- = -\csc u \, .\cot u. \, du/dx$

Example

$$xy + \sin y = 0$$

$$(x\frac{dy}{dx} + y) + \cos y.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{x + \cos y}$$

Example

Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x=\pi/3$.

Solution

$y^- = 5\sin^4 x \, . \cos x$

the tangent line has slope

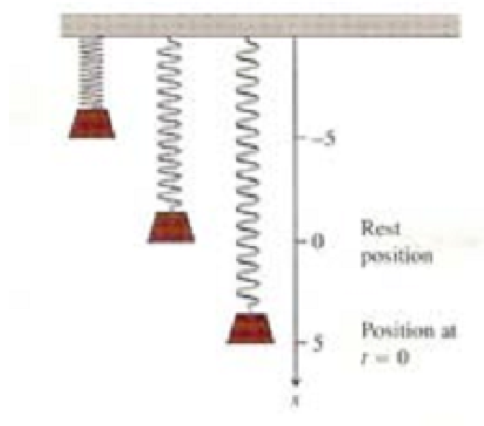
$$\frac{dy}{dx}\Big|_{x=\frac{\pi}{4}} = \frac{45}{32}$$

Example

A body hanging from a spring is starched 5 units beyond its rest position and released at time t=0 to bob up and down. Its position at any later time is

s = 5 cost

What are its velocity and acceleration at time t?



Solution

Position s = 5 cost

Velocity v(t) = -5sint

Acceleration a(t) = -5cost

H.W

Find the derivative for

$$y = \frac{\sin x}{x}$$

$$y = \frac{1}{\sin x}$$

$$y^2 = x^2 + \sin xy$$

2- Find the second derivative of $y = \sec^2 5x$

THEOREM: We have

(a) $(\sin^{-1} u)' = \dfrac{1}{\sqrt{1-u^2}}u'$	(d) $(\cot^{-1} u)' = -\dfrac{1}{1+u^2}u'$
(b) $(\cos^{-1} u)' = -\dfrac{1}{\sqrt{1-u^2}}u'$	(e) $(\sec^{-1} u)' = \dfrac{1}{u\sqrt{u^2-1}}u'$
(c) $(\tan^{-1} u)' = \dfrac{1}{1+u^2}u'$	(f) $(\csc^{-1} u)' = -\dfrac{1}{u\sqrt{u^2-1}}u'$

Proof:

(a) Let $y = \sin^{-1} u$, then $\sin y = u$. Therefore

$$(\sin y)' = u' \quad \Longrightarrow \quad \cos y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\cos y}$$

Since $-\dfrac{\pi}{2} \le \underbrace{\sin^{-1} u}_{y} \le \dfrac{\pi}{2}$, it follows that $\cos y \ge 0$. Hence

$$\cos y = \sqrt{1-\sin^2 y} = [\sin y = u] = \sqrt{1-u^2} \quad \Longrightarrow \quad y' = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}$$

(b) Let $y = \cos^{-1} u$, then $\cos y = u$. Therefore

$$(\cos y)' = u' \quad \Longrightarrow \quad -\sin y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\sin y}$$

Since $0 \le \underbrace{\cos^{-1} u}_{y} \le \pi$, it follows that $\sin y \ge 0$. Hence

$$\sin y = \sqrt{1-\cos^2 y} = [\cos y = u] = \sqrt{1-u^2} \quad \Longrightarrow \quad y' = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}$$

(c) Let $y = \tan^{-1} u$, then $\tan y = u$. Therefore

$$(\tan y)' = u' \quad \Longrightarrow \quad \sec^2 y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec^2 y}$$

Note, that $\sec^2 y = 1 + \tan^2 y = [\tan y = u] = 1 + u^2$. Hence

$$y' = \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}$$

(d) Let $y = \cot^{-1} u$, then $\cot y = u$. Therefore

$$(\cot y)' = u' \quad \Longrightarrow \quad -\csc^2 y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\csc^2 y}$$

Note, that $\csc^2 y = 1 + \cot^2 y = [\cot y = u] = 1 + u^2$. Hence

$$y' = -\frac{u'}{\csc^2 y} = -\frac{u'}{1+u^2}$$

(e) Let $y = \sec^{-1} u$, then $\sec y = u$. Therefore

$$(\sec y)' = u' \quad \Longrightarrow \quad \sec y \tan y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec y \tan y}$$

Since $\underbrace{\sec^{-1} u}_{y} \in [0, \pi/2) \cup [\pi, 3\pi/2)$, it follows that $\tan y \geq 0$. Hence

$$\sec y \tan y = \sec y \sqrt{\sec^2 y - 1} = [\sec y = u] = u\sqrt{u^2-1} \quad \Longrightarrow \quad y' = \frac{u'}{\sec y \tan y} = \frac{u'}{u\sqrt{u^2-1}}$$

(f) Let $y = \csc^{-1} u$, then $\csc y = u$. Therefore

$$(\csc y)' = u' \quad \Longrightarrow \quad -\csc y \cot y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\csc y \cot y}$$

Since $\underbrace{\csc^{-1} u}_{y} \in (0, \pi/2] \cup (\pi, 3\pi/2]$, it follows that $\cot y \geq 0$. Hence

$$\csc y \cot y = \csc y \sqrt{\csc^2 y - 1} = [\csc y = u] = u\sqrt{u^2-1} \quad \Longrightarrow \quad y' = -\frac{u'}{\csc y \cot y} = -\frac{u'}{u\sqrt{u^2-1}}$$

EXAMPLES:

(a) Let $f(x) = x\tan^{-1}(1-2x)$. Find $f'(x)$.

(b) Let $f(x) = 2^{\sin^{-1}(4x)}$. Find $f'(x)$.

(c) Let $f(x) = \sqrt{\sec^{-1}(1-3x)}$. Find $f'(x)$.

EXAMPLES:

(a) Let $f(x) = x\tan^{-1}(1-2x)$. Then

$$
\begin{aligned}
f'(x) = [x\tan^{-1}(1-2x)]' &= x'\tan^{-1}(1-2x) + x[\tan^{-1}(1-2x)]' \\
&= 1\cdot\tan^{-1}(1-2x) + x\frac{1}{1+(1-2x)^2}\cdot(1-2x)' \\
&= \tan^{-1}(1-2x) + x\frac{1}{1+(1-2x)^2}\cdot(-2) \\
&= \tan^{-1}(1-2x) - \frac{2x}{1+(1-2x)^2} \\
&= \tan^{-1}(1-2x) - \frac{x}{1-2x+2x^2}
\end{aligned}
$$

(b) Let $f(x) = 2^{\sin^{-1}(4x)}$. Then

$$
\begin{aligned}
f'(x) = \left[2^{\sin^{-1}(4x)}\right]' &= 2^{\sin^{-1}(4x)}\ln 2\cdot\left[\sin^{-1}(4x)\right]' \\
&= 2^{\sin^{-1}(4x)}\ln 2\frac{1}{\sqrt{1-(4x)^2}}\cdot(4x)' \\
&= 2^{\sin^{-1}(4x)}\ln 2\frac{1}{\sqrt{1-(4x)^2}}\cdot 4 \\
&= \frac{2^{\sin^{-1}(4x)+2}\ln 2}{\sqrt{1-(4x)^2}}
\end{aligned}
$$

(c) Let $f(x) = \sqrt{\sec^{-1}(1-3x)}$. Then

$$
\begin{aligned}
f'(x) = [(\sec^{-1}(1-3x))^{1/2}]' &= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\cdot[\sec^{-1}(1-3x)]' \\
&= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\frac{1}{(1-3x)\sqrt{(1-3x)^2-1}}\cdot(1-3x)' \\
&= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\frac{1}{(1-3x)\sqrt{(1-3x)^2-1}}\cdot(-3) \\
&= -\frac{3}{2(1-3x)\sqrt{3x(3x-2)\sec^{-1}(1-3x)}}
\end{aligned}
$$

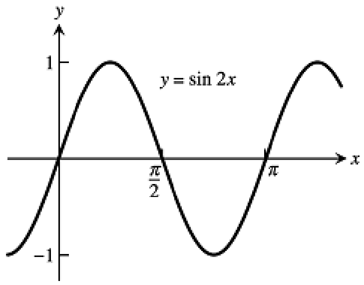
Solved question

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$ **14.** $\sin(x/2)$

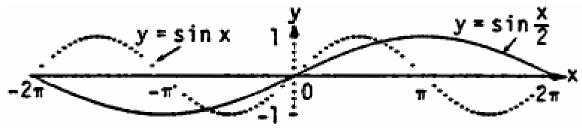
Solution

13.



period $= \pi$

14.



period $= 4\pi$

15. $\cos \pi x$ **16.** $\cos \dfrac{\pi x}{2}$

17. $-\sin \dfrac{\pi x}{3}$ **18.** $-\cos 2\pi x$

19. $\cos\left(x - \dfrac{\pi}{2}\right)$ **20.** $\sin\left(x + \dfrac{\pi}{2}\right)$

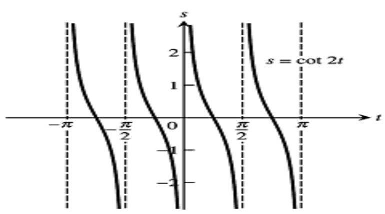
21. $\sin\left(x - \dfrac{\pi}{4}\right) + 1$ **22.** $\cos\left(x + \dfrac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

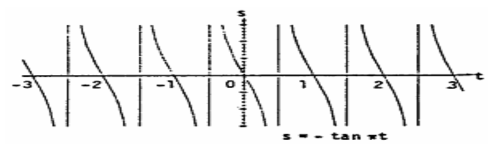
23. $s = \cot 2t$ **24.** $s = -\tan \pi t$

Solution

23. period $= \frac{\pi}{2}$, symmetric about the origin



24. period $= 1$, symmetric about the origin

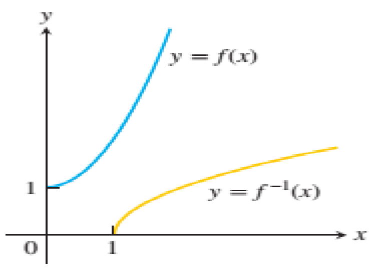


25. $s = \sec\left(\frac{\pi t}{2}\right)$ **26.** $s = \csc\left(\frac{t}{2}\right)$

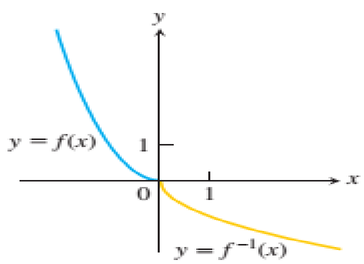
Inverse trigonometric

Each of Exercises 13–18 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

13. $f(x) = x^2 + 1, \quad x \geq 0$



14. $f(x) = x^2, \quad x \leq 0$

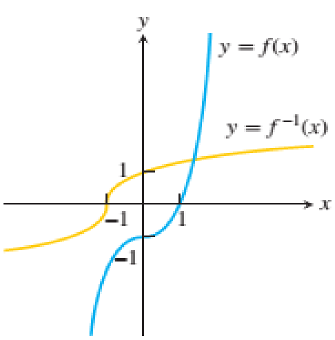


Solution

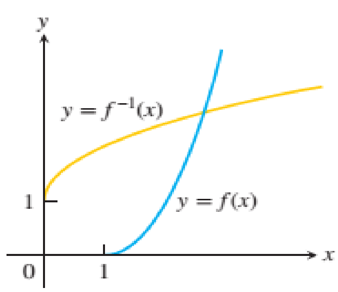
13. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}$
    Step 2: $y = \sqrt{x-1} = f^{-1}(x)$

14. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \leq 0$.
    Step 2: $y = -\sqrt{x} = f^{-1}(x)$

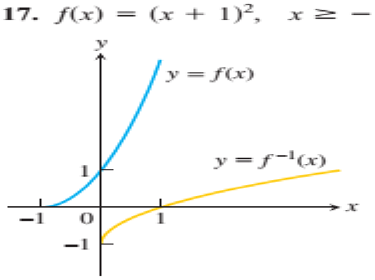
15. $f(x) = x^3 - 1$



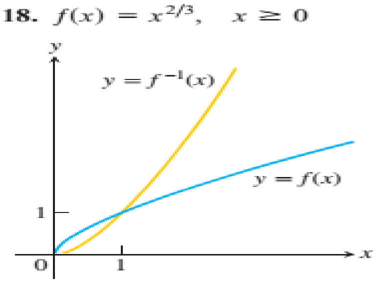
16. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



17. $f(x) = (x+1)^2, \quad x \geq -1$



18. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

19. $f(x) = x^5$ **20.** $f(x) = x^4, \quad x \geq 0$

21. $f(x) = x^3 + 1$ **22.** $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$ **24.** $f(x) = 1/x^3, \quad x \neq 0$

19. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$
Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left(x^{1/5}\right)^5 = x$ and $f^{-1}(f(x)) = (x^5)^{1/5} = x$

20. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$
Step 2: $y = \sqrt[4]{x} = f^{-1}(x)$;
Domain of f^{-1}: $x \geq 0$, Range of f^{-1}: $y \geq 0$;
$f(f^{-1}(x)) = \left(x^{1/4}\right)^4 = x$ and $f^{-1}(f(x)) = (x^4)^{1/4} = x$

21. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$
Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = \left((x^3+1) - 1\right)^{1/3} = (x^3)^{1/3} = x$

In Exercises 25–28:

a. Find $f^{-1}(x)$.

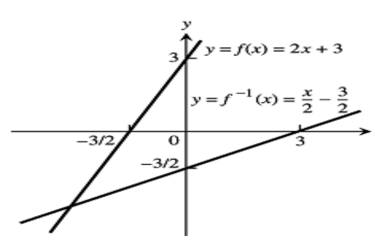
b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$ **26.** $f(x) = (1/5)x + 7, \quad a = -1$

25. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$

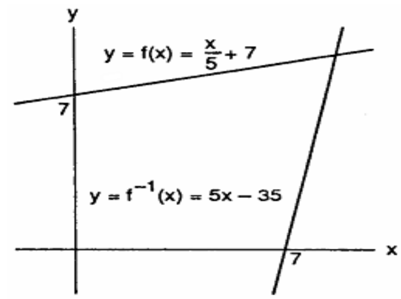
(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2, \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

26. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$

(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5}, \left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

27. $f(x) = 5 - 4x, \quad a = 1/2$ **28.** $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

راجع 7.7

Chapter7

Matrices and determinants

Matrices

Definition: An m×n matrix is a rectangular array of numbers (m rows and n columns) enclosed in brackets. The numbers are called the elements of the matrix.

Examples:

(i) A 2×3 matrix has 2 rows and 3 columns:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}$$

(ii) Here's a 3×3 **square** matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$$

(iii) **Column vectors** are matrices with only one column:

$$\mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}$$

(iv) **Row vectors** are matrices which only have one row:

$$\mathbf{b} = (1\ 2\ 3)\ .$$

A general real matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \times n$ elements is of the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \qquad (1)$$

4×4 و، 3×3 ، 2×2

$$1)\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \qquad 2)\begin{vmatrix} 1 & 3 & 9 \\ 2 & 6 & 3 \\ 8 & 7 & 4 \end{vmatrix} \qquad 3)\begin{vmatrix} -1 & 2 & 1 & 0 \\ -3 & 1 & 1 & 5 \\ -2 & 2 & 5 & 1 \\ 0 & 2 & 0 & -3 \end{vmatrix}$$

We refer to the elements via double indices as follows

(i) The first index represents the row. (ii) The second index represents the column.

(a_{32}) is the element in **row 3**, **column 2** of the **matrix A**.

Use lowercase boldface (or underlined) letters **for vectors**

a b c (or a, b, c)

Use uppercase boldface (or underlined) letters for **matrices**

A B C (or A, B , C)

Refer to the respective elements by lowercase letters with the

appropriate number of indices e.g.

bi is a vector element

aij is a matrix element

Matrix algebra

Two matrices are **equal** if they have the same size and if their corresponding elements are identical , i.e.

$$\mathbf{A} = \mathbf{B}$$

if and only if

$$a_{ij} = b_{ij} \quad \text{for } i = 1, \ldots, m; \; j = 1, \ldots, n$$

Matrix addition

- ❖ Two matrices can only be added if they have the same size.
- ❖ The result is another matrix of the same size.

- We add matrices by adding their corresponding elements, i.e.

$$A = B + C$$

is obtained (element-wise) via

$a_{ij} = b_{ij} + c_{ij}$ for $i = 1, ..., m; j = 1, ..., n$

Problem 1. Add the matrices

(a) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$

(a) Adding the corresponding elements gives:

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 2+(-3) & -1+0 \\ -7+7 & 4+(-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

(b) Adding the corresponding elements gives:

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 3+2 & 1+7 & -4+(-5) \\ 4+(-2) & 3+1 & 1+0 \\ 1+6 & 4+3 & -3+4 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{5} & \mathbf{8} & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \mathbf{7} & \mathbf{7} & \mathbf{1} \end{pmatrix}$$

Problem 2. Subtract

(a) $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$ from $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$ from $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$

(a) $$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 2-(-3) & -1-0 \\ -7-7 & 4-(-4) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{-14} & \mathbf{8} \end{pmatrix}$$

(b) $$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 1-7 & -4-(-5) \\ 4-(-2) & 3-1 & 1-0 \\ 1-6 & 4-3 & -3-4 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & \mathbf{-6} & \mathbf{1} \\ \mathbf{6} & \mathbf{2} & \mathbf{1} \\ \mathbf{-5} & \mathbf{1} & \mathbf{-7} \end{pmatrix}$$

Problem 3. If

$A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and

$C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find $A + B - C$.

$$A + B = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

(from Problem 1)

Hence, $A + B - C = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$

$$= \begin{pmatrix} -1 - 1 & -1 - 0 \\ 0 - (-2) & 0 - (-4) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{-2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$$

Alternatively $A + B - C$

$$= \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 2 - 1 & 0 + (-1) - 0 \\ 7 + (-7) - (-2) & -4 + 4 - (-4) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{-2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix} \text{ as obtained previously}$$

Multiplication

When a matrix is multiplied by a number, called scalar multiplication, a single matrix results in which each element of the original matrix has been multiplied by the number.

Problem 4. If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find $2A - 3B + 4C$.

$$2A = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix},$$

$$3B = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix},$$

$$\text{and} \quad 4C = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

Hence $2A - 3B + 4C$

$$= \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} -6-6+4 & 0-(-3)+0 \\ 14-(-21)+(-8) & -8-12+(-16) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{-8} & \mathbf{3} \\ \mathbf{27} & \mathbf{-36} \end{pmatrix}$$

EX.

$$A=\begin{pmatrix} 30 & -40 & 90 \\ -25 & -80 & 100 \end{pmatrix} \qquad B=\begin{pmatrix} 10 & 15 \\ 70 & 90 \\ 120 & 110 \end{pmatrix}$$

Find

1) AB 2) BA

$$1)\ AB=\begin{pmatrix} (30 \quad -40 \quad 90)\begin{pmatrix} 10 \\ 70 \\ 120 \end{pmatrix} & (30 \quad -40 \quad 90)\begin{pmatrix} 15 \\ 90 \\ 110 \end{pmatrix} \\ (-25 \quad -80 \quad 100)\begin{pmatrix} 10 \\ 70 \\ 120 \end{pmatrix} & (-25 \quad -80 \quad 100)\begin{pmatrix} 15 \\ 90 \\ 110 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 8300 & 6750 \\ 6150 & 3425 \end{pmatrix}$$

2) $BA=\begin{pmatrix} (10\quad 15)\begin{pmatrix}30\\-25\end{pmatrix} & (10\quad 15)\begin{pmatrix}-40\\-80\end{pmatrix} & (10\quad 15)\begin{pmatrix}90\\100\end{pmatrix} \\ (70\quad 90)\begin{pmatrix}30\\-25\end{pmatrix} & (70\quad 90)\begin{pmatrix}-40\\-80\end{pmatrix} & (70\quad 90)\begin{pmatrix}90\\100\end{pmatrix} \\ (120\quad 110)\begin{pmatrix}30\\-25\end{pmatrix} & (120\quad 110)\begin{pmatrix}-40\\-80\end{pmatrix} & (120\quad 110)\begin{pmatrix}90\\100\end{pmatrix} \end{pmatrix}=\begin{pmatrix}-75 & -1600 & 2400\\ -150 & -10000 & 15300\\ 850 & -13600 & 21800\end{pmatrix}$

Problem 5. If $A=\begin{pmatrix}2 & 3\\ 1 & -4\end{pmatrix}$ and $B=\begin{pmatrix}-5 & 7\\ -3 & 4\end{pmatrix}$ find $A\times B$.

Let $A\times B=C$ where $C=\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}$

C_{11} is the sum of the products of the first row elements of A and the first column elements of B taken one at a time,

$$C_{11}=(2\times(-5))+(3\times(-3))=-19$$

$$C_{12}=(2\times 7)+(3\times 4)=26$$

$$C_{21}=(1\times(-5))+(-4)\times(-3))=7$$

$$C_{22}=(1\times 7)+((-4)\times 4)=-9$$

Thus, $A \times B = \begin{pmatrix} -19 & 26 \\ 7 & -9 \end{pmatrix}$

Problem 6. Simplify 3x3 * 1x3 =3x1

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (3 \times 2) + (4 \times 5) + (0 \times (-1)) \\ (-2 \times 2) + (6 \times 5) + (-3 \times (-1)) \\ (7 \times 2) + (-4 \times 5) + (1 \times (-1)) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{26} \\ \mathbf{29} \\ \mathbf{-7} \end{pmatrix}$$

Problem 7. If $A = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}$ find $A \times B$.

3 x 3 * 2 x 3 = 3x2

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} [(3\times 2) + (4\times 5) + (0\times(-1))] & [(3\times(-5)) + (4\times(-6)) + (0\times(-7))] \\ [(-2\times 2) + (6\times 5) + (-3\times(-1))] & [(-2\times(-5)) + (6\times(-6)) + (-3\times(-7))] \\ [(7\times 2) + (-4\times 5) + (1\times(-1))] & [(7\times(-5)) + (-4\times(-6)) + (1\times(-7))] \end{pmatrix} = \begin{pmatrix} 26 & -39 \\ 29 & -5 \\ -7 & -18 \end{pmatrix}$$

Problem 8. Determine

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

Let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ be two column vectors with n real elements each:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}.$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n .$$

$$\mathbf{a}^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix}.$$

$$\mathbf{a}^T \mathbf{b} \stackrel{\text{def}}{=} \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

Problem 9. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ show that $A \times B \neq B \times A$.

$$A \times B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} [(2 \times 2) + (3 \times 0)] & [(2 \times 3) + (3 \times 1)] \\ [(1 \times 2) + (0 \times 0)] & [(1 \times 3) + (0 \times 1)] \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} [(2 \times 2) + (3 \times 1)] & [(2 \times 3) + (3 \times 0)] \\ [(0 \times 2) + (1 \times 1)] & [(0 \times 3) + (1 \times 0)] \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix}$ then $\boldsymbol{A \times B \neq B \times A}$

Determinant of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is written $\det \mathbf{A}$ or $|\mathbf{A}|$ or

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The determinant of a 3×3 matrix is written as

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The sign of a minor depends on its position within the matrix, the sign pattern being $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

That is the 3×3 determinant is defined in terms of determinants of 2×2 sub-matrices of A. These are called the minors of A.

Problem 10. Determine the value of $\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix}$

$$\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix} = (3 \times 4) - (-2 \times 7) \qquad = 12 - (-14) = \mathbf{26}$$

Problem 15. Evaluate $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

Using the first row: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & 6 \\ -4 & 2 \end{vmatrix} - 4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + (-3)\begin{vmatrix} -5 & 2 \\ -1 & -4 \end{vmatrix}$$
$$= (4 + 24) - 4(-10 + 6) - 3(20 + 2)$$
$$= 28 + 16 - 66 = \mathbf{-22}$$

Using the second column: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= -4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -3 \\ -5 & 6 \end{vmatrix}$$
$$= -4(-10 + 6) + 2(2 - 3) + 4(6 - 15)$$
$$= 16 - 2 - 36 = \mathbf{-22}$$

Calculate the determinant of

$$\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$$

Calculate the determinant of

$$\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$$

Calculate the determinant of

$$\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix} \qquad [-242.83]$$

Points to note:

- <u>the determinant $\det \mathbf{A}$ is equal to zero</u> if
  - (i) rows or columns of $\mathbf{A}$ are multiples of each other,
  - (ii) rows or columns are linear combinations of each other,
  - (iii) entire rows or columns are zero;

  if $\det \mathbf{A} = 0$ **the matrix A is called a <u>singular</u> matrix**;
- for any square matrices $\mathbf{A}$ and $\mathbf{B}$ there holds

$$\det \mathbf{A} = \det(\mathbf{A}^T), \qquad \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

- for the unit matrix $\mathbf{I}$ one has $\det \mathbf{I} = 1$.

THE INVERSE OR RECIPROCAL OF A 2 BY 2 MATRIX

The inverse of matrix A is A^{-1} such that $A \times A^{-1} = I$, the unit matrix.

Let matrix A be $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and let the inverse matrix, A^{-1} be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, since $A \times A^{-1} = I$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying the matrices on the left hand side, gives

$$\begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating corresponding elements gives:

$$b + 2d = 0, \quad \text{i.e.} \quad b = -2d$$

and $\quad 3a + 4c = 0, \quad$ i.e. $\quad a = -\frac{4}{3}c$

Substituting for a and b gives:

$$\begin{pmatrix} -\frac{4}{3}c + 2c & -2d + 2d \\ 3\left(-\frac{4}{3}c\right) + 4c & 3(-2d) + 4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e.
$$\begin{pmatrix} \frac{2}{3}c & 0 \\ 0 & -2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

showing that $\frac{2}{3}c = 1$, i.e. $c = \frac{3}{2}$ and $-2d = 1$, i.e. $d = -\frac{1}{2}$

Since $b = -2d$, $b = 1$ and since $a = -\frac{4}{3}c$, $a = -2$.

Thus the inverse of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that is,

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example. To find

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1}$$

first check that

$$\det\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 5 = 1$$

Then

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

Note: the inverse $\mathbf{A}^{-1}$ exists if (and only if) $\det \mathbf{A} \neq 0$.

Problem 13. Determine the inverse of

$$\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} = \frac{1}{(3 \times 4) - (-2 \times 7)}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ \frac{-7}{26} & \frac{3}{26} \end{pmatrix}$$

Determine the inverse of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$

$$\left[\begin{pmatrix} \frac{7}{17} & \frac{1}{17} \\ \frac{4}{17} & \frac{3}{17} \end{pmatrix}\right]$$

Determine the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$

$$\left[\begin{pmatrix} 7\frac{5}{17} & 8\frac{4}{7} \\ -4\frac{2}{7} & -6\frac{3}{7} \end{pmatrix}\right]$$

THE INVERSE OR RECIPROCAL OF A 3 BY 3 MATRIX

The **inverse of matrix A, A^{-1}** is given by

$$A^{-1} = \frac{\text{adj}\,A}{|A|}$$

where adj A is the adjoint of matrix A and $|A|$ is the determinant of matrix A.

Problem 17. Determine the inverse of the matrix $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$

The inverse of matrix A, $A^{-1} = \dfrac{\text{adj}\,A}{|A|}$

The adjoint of A is found by:

(i) obtaining the matrix of the cofactors of the elements, and

(ii) transposing this matrix.

The cofactor of element 3 is $+\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 21$.

The cofactor of element 4 is $-\begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} = 11$, and so on.

The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The transpose of the matrix of cofactors, i.e. the adjoint of the matrix, is obtained by writing the rows as columns, and is $\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$

From Problem 14, the determinant of

$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$ is 113.

Hence the inverse of $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}{113} \text{ or } \mathbf{\frac{1}{113}\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}$$

Problem 18. Find the inverse of

$$\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$$

$$\text{Inverse} = \frac{\text{adjoint}}{\text{determinant}}$$

The matrix of cofactors is $\begin{pmatrix} -17 & 9 & 15 \\ 23 & -13 & -21 \\ 18 & -10 & -16 \end{pmatrix}$

$$\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}$$

The determinant of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= 1(7-24) - 5(-21+12) - 2(18-3)$$

$$= -17 + 45 - 30 = -2$$

Hence the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= \frac{\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}}{-2}$$

$$= \begin{pmatrix} \mathbf{8.5} & \mathbf{-11.5} & \mathbf{-9} \\ \mathbf{-4.5} & \mathbf{6.5} & \mathbf{5} \\ \mathbf{-7.5} & \mathbf{10.5} & \mathbf{8} \end{pmatrix}$$

1. Write down the transpose of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[\begin{pmatrix} 4 & -2 & 5 \\ -7 & 4 & 7 \\ 6 & 0 & -4 \end{pmatrix}\right]$$

Write down the transpose of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

$$\left[\begin{pmatrix} 3 & 5 & -1 \\ 6 & -\frac{2}{3} & 0 \\ \frac{1}{2} & 7 & \frac{3}{5} \end{pmatrix}\right]$$

Determine the adjoint of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

$$\left[\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -12 \\ -34 & -63 & 2 \end{pmatrix}\right]$$

Determine the adjoint of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

$$\left[\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}\right]$$

Find the inverse of $\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$

$$\left[-\frac{1}{212}\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -12 \\ -34 & -63 & 2 \end{pmatrix}\right]$$

Find the inverse of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

$$\left[-\frac{15}{923}\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}\right]$$